PHY321 Homework Set 11

1. [10 pts] Consider the central-force problem for a potential $U(r) = -k/r^2$ and a reduced mass μ. Here, k is a positive constant.

   (a) Sketch the effective potential for large and small values of angular momentum $|\ell|$. Is there a difference in the sign of the potential? Use your graph and energy conservation and show that for large ℓ the orbits extend from some minimal value of radius r_{min} up to infinity. Obtain r_{min} as a function of energy E. Show that for small ℓ the orbits reach $r = 0$, but can either extend up to a finite r_{max} or up to infinity, depending on E. What value of $|\ell| = \ell_s$ separates the low and high value regions of ℓ, with different types of orbits? Obtain r_{max} as a function of E.

   (b) Turn to the orbit equation and find general analytic solutions $r(\theta)$ for high and low values of $|\ell|$. (Note that, $\ell = \pm \ell_s$ requires a separate solution, but you are not asked to investigate it here.)

   (c) As an example, adjust the arbitrary constants in the solution $r(\theta)$ for *high* $|\ell|$, so that the orbit is symmetric about the axis $\theta = 0$ and that the orbit extends from r_{min} on. Note that at r_{min}, $\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{r} = 0$. Sketch the orbit in the plane of motion. What are the asymptotic directions at $r \to \infty$, in terms of angle, for the orbit?

   (d) As another example, adjust the arbitrary constants in the solution $r(\theta)$ for low $|\ell|$, so that the orbit is symmetric about the axis $\theta = 0$ and that it extends up to r_{max}. Again, at r_{max}, $\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{r} = 0$. Sketch the trajectory in the plane of motion for $\theta > 0$.

2. [5 pts] Consider a particle of mass m moving under the influence of a central force

$$F = -\frac{k}{r^2} + \frac{c}{r^3},$$

   where k and c are positive constants.

   (a) Demonstrate that the solution to the orbit equation can be put into the form

$$\frac{\alpha}{r} = 1 + \epsilon \cos(\nu\theta),$$

   which is an ellipse for $\epsilon < 1$ and $\nu = 1$. How are ν and α related to c, k and ℓ? What happens to ν when c approaches zero?

   (b) Discuss the character of the orbit when $\epsilon < 1$ and $\nu \neq 1$. Hint: Consider the change in the angle after one period in r, e.g. when reaching again $r = r_{\text{min}}$. Sketch the orbit assuming a small deviation of ν from 1. Note: Effects of general relativity contribute minute corrections to $F = -k/r^2$ for planets. Those corrections are characterized by a faster fall-off than $1/r^2$, as in the present problem, and they cause modifications of the orbits compared to elliptical.

3. [5 pts] Use of characteristics of the Earth's orbit as a reference, and specifically of the period and of the astronomical unit (1 AU) equal to Earth-Sun semimajor axis, facilitates considerations of different orbits within the solar system. As an example consider the case of the Halley's comet which moves about the Sun on an elliptical orbit with an eccentricity of $\epsilon = 0.967$ and with a period of 76 years.

   (a) Using the above, calculate the distance in astronomical units of the comet from the Sun at perihelion and at aphelion.

   (b) Find the aerial velocity in AU^2/y for the comet. How does that velocity compare with the aerial velocity for Earth?

   (c) Find the linear velocity of the comet when it crosses the Earth's orbit, both in AU/y and in m/s. Compare that velocity to the velocity of Earth, both in AU/y and m/s. For this, take the Earth's orbit as approximately circular.

4. [10 pts] A satellite moves in an elliptical orbit around Earth.

   (a) If the ratio of the maximum angular velocity to the minimum angular velocity for the satellite is 3.4, what is the eccentricity ϵ of the orbit? Hint: Use angular momentum conservation.

   (b) If the perigee of the orbit is 300 km above the Earth's surface, how high is the apogee above the surface?

   (c) What are the semimajor and semiminor axes of the orbit?

   (d) What is the period of the orbit?

   (e) What are the linear and angular velocities of the satellite at the perigee and at the apogee? Hint: Start with aerial velocity.

   (f) If the satellite were to be slowed down so that it could reach the Earth's surface, would it be better to do it at the perigee or apogee? Calculate the required change in linear velocity Δv for both locations. Hint: Consider change in energy associated with change in semimajor axis.

   (g) If the exhaust velocity for satellite boosters is $u = 3100\,\mathrm{m/s}$, what fraction of the satellite mass would need to be burnt up as fuel to arrive at the desired change in velocity for the more convenient of the locations? For simplicity assume that the boosters operate over such a short time that effects of gravitational and centrifugal forces can be ignored.

5. [5 pts] Consider objects of mass m moving under the influence of a central gravitational force characterized by potential energy $U = -k/r$.

   (a) Demonstrate that when two orbits, circular and parabolic, have the same net angular momentum, the parabolic orbit has a perihelion that is half the radius of the circular orbit.

   (b) The speed of a particle at any point of a parabolic orbit is larger by a factor of $\sqrt{2}$ than the speed of particle on a circular orbit passing through the same point.

6. [5 pts] At perihelion, a particle of mass m in an elliptical orbit in a gravitational $1/r$ potential receives an impulse $m\,\Delta v$ in the radial direction. Find the semi-major axis a_{new} and eccentricity ϵ_{new} of the new elliptical orbit in terms of the old ellipse's parameters a_{old} and ϵ_{old}. Are these parameters increasing or decreasing in effect of the impulse. Does the direction of the impulse, inward or outward, matter for those specific parameters?