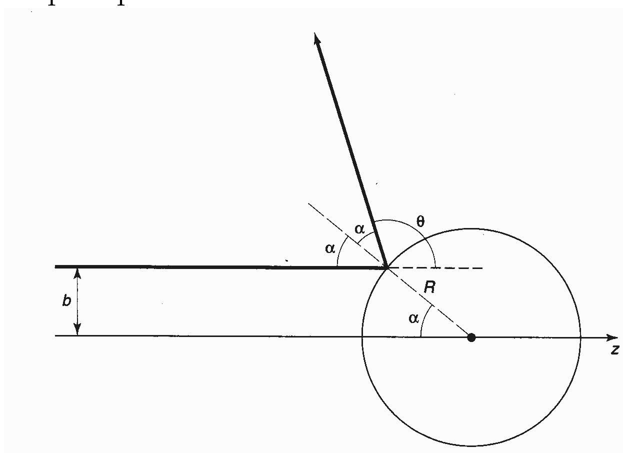
PHY321 Homework Set 5

1. [5 pts] Determine the differential cross section $\sigma(\theta) \equiv \mathrm{d}\sigma/\mathrm{d}\Omega$ and total cross section σ_t for elastic scattering of a point particle from a strong repulsive potential sphere of radius R:

$$U(r) = \begin{cases} 0, & r < R\,, \\ U_0, & r > R\,, \end{cases}$$

where $U_0 \to \infty$. For a particle scattered from a strong potential, the law of reflection is valid, see the figure. Hint: From geometry work out the relation between the scattering angle θ and the impact parameter b.



2. [10 pts] Start with the Rutherford scattering cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'} = \frac{1}{16}\left(\frac{kq_1\,q_2}{T'}\right)^2 \frac{1}{\sin^4(\theta'/2)}\,,$$

where $k = 1/4\pi\epsilon_0$ and T' is kinetic energy within the center of mass of the charges q_1 and q_2.

   (a) By integrating the above differential cross section, obtain the net cross section for scattering to angles $\theta' > \theta_0$:

$$\sigma_{\theta'>\theta_0} = \int_{\theta'>\theta_0} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'}\,\mathrm{d}\Omega'\,.$$

   Hint: When a solid-angle integration extends over the full range of azimuthal angle, and there is no dependence on the latter angle in the subintegral function, then the suitable element of solid-angle integration is $\mathrm{d}\Omega' \equiv 2\pi \sin\theta' \mathrm{d}\theta'$. In the particular case, it may be further useful to change the variable of integration from θ' to $\sin(\theta'/2)$.

   (b) Find the net cross sections for Rutherford scattering of 7.7 MeV alpha particles from gold nuclei, through angles I: θ' greater than 170°, II: greater than 90°, and III: greater than 10°. Express your cross sections both in m^2 and in barns; $1\,\mathrm{b} = 10^{-28}\,\mathrm{m}^2 = 100\,\mathrm{fm}^2$ represents approximately the size of uranium nucleus. *Data:* $m_\alpha = 4\,\mathrm{u}$, $Z_\alpha = 2$, $m_{\mathrm{Au}} = 197\,\mathrm{u}$ and $Z_{\mathrm{Au}} = 79$; $e^2/4\pi\epsilon_0 = 1.4400\,\mathrm{MeV\,fm}$. For $m_\alpha \ll m_{\mathrm{Au}}$, you can in practice ignore the difference between laboratory and center-of-mass reference frames.

3. [5 pts] A rocket of mass m_0 starts its engine in interstellar space. Assuming constant speed u of the exhaust gas relative to the rocket, at what fraction of the original mass is the rocket going to achieve maximal momentum?

4. [10 pts] Consider the problem of a rocket ascending vertically against gravity. The rocket starts from rest and its initial mass is m_0. The rocket's fuel burns at a constant rate α and exhaust gas leaves the rocket at a constant speed u relative to the rocket. A convenient characteristic of a rocket, that is commonly used in place of α, is the initial trust-to-weight ratio $\tau_0 = \alpha\, u / m_0\, g$.

   (a) From the expression for the velocity of the rocket as a function of the remaining mass m,

   $$v = -(m_0 - m)\,\frac{g}{\alpha} + u \ln\left(\frac{m_0}{m}\right),$$

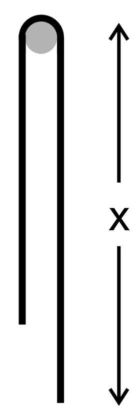
   eliminate α and write the velocity in terms of u, the mass ratio m_0/m and τ_0.

   (b) Demonstrate that, for the lift-off to occur, the rocket must be light enough so that $\tau_0 > 1$.

   (c) Integrate the velocity with respect to time to obtain elevation h of the rocket as a function of the remaining mass m. Note that the integration with respect to time can be easily converted to integration with respect to mass exploiting the linear relation between the two variables. Note further that $\int dx \ln x = x \ln x - x$. Again eliminate α from your result and represent h in terms of τ_0, u, g and the mass ratio m_0/m.

   (d) Consider the case of Ariane 5 rocket with initial mass of $m_0 = 7.77 \times 10^5\,\text{kg}$. During the initial stage-0 of the flight, boosters are used that provide a thrust of $\alpha u = 1.29 \times 10^7\,\text{N}$ and employ solid fuel with exhaust velocity of $u = 3010\,\text{m/s}$. At the end of stage-0, the rocket mass drops to $m = 2.23 \times 10^5\,\text{kg}$. Find τ_0 and mass ratio m_0/m. Use those to determine the expected velocity and elevation of the rocket at the end of stage-0.

5. [10 pts] A uniform rope of mass M and length L is hung off a small peg. The rope can slide without friction over the peg.

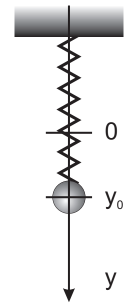


   (a) With x denoting the length of rope hanging to the right of the peg, obtain an equation of motion for $x(t)$.

   (b) Solve the equation of motion, if an x_0-stretch of the rope hangs to the right of the peg at $t = 0$ and the rope is then at rest. How does the fate of the rope depend on whether $x_0 > L/2$ or $x_0 < L/2$?

   (c) Find the time t_f that it takes for the rope to fall off the peg. How does this time depend on M?

6. [5 pts] A massless spring hangs down from a support, with its lower end at $y = 0$, where the y-axis is vertical and points down. When a small unknown mass is attached to the spring, the lower end of the spring moves down to a position y_0 for the mass being in equilibrium.



(a) Demonstrate that when the mass is pulled down to a position $y = y_0 + A$ and released from rest, it will execute a simple harmonic motion around y_0.

(b) Express the period of oscillations of the mass in terms of y_0 and g.