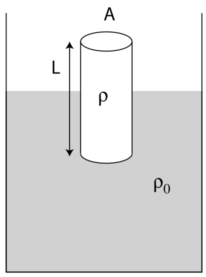
PHY321 Homework Set 6

1. [5 pts] [adapted from Graduate School qualifying exam] A large axially symmetric cylinder of length L, cross-sectional area A and of average density ρ is floating with its axis vertical, in a fluid of density ρ_0, $\rho_0 > \rho$.



   (a) Determine the frequency ν_0 of small-amplitude vertical oscillations of the cylinder. Note: Effects of viscosity and fluid adhesion and cohesion may be ignored.

   (b) Compute the frequency for $\rho_0 = 1.00\,\text{g/cm}^3$, $\rho = 0.80\,\text{g/cm}^3$ and $L = 3.0\,\text{cm}$.

2. [10 pts] Two masses m_1 and m_2, connected by a massless spring of neutral length x_0, move freely along the x-axis. For small changes in spring's length, the force produced by the spring is proportional to the change in length, with the coefficient of proportionality k.

   (a) Rely on the third Newton's law and demonstrate directly that the center of mass for m_1 and m_2 moves at constant velocity.

   (b) Further, rely on the third Newton's law and demonstrate that for separations close to x_0, the masses execute a simple harmonic motion in their separation. Obtain the angular frequency ω_0 of small oscillations of the masses in relative distance.

3. [5 pts] A simple harmonic oscillator consists of a $m = 15\,\text{g}$ mass attached to a spring with spring constant of $k = 4.0\,\text{N/m}$. The mass is displaced by $A = 3\,\text{cm}$ and released from rest.

   (a) Determine the anticipated natural frequency ν_0 and period τ_0 for the motion,

   (b) the total energy,

   (c) and the anticipated maximum speed.

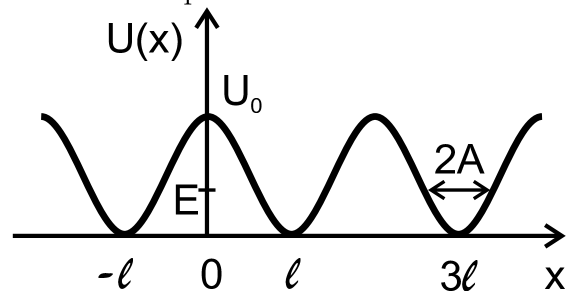
   (d) In the actual measurement of the system, the amplitude of the oscillations is found to decrease to half of the original value after 8.0 s. Determine the parameter β for this motion.

   (e) Find the frequency ν_1 for the damped motion and compare it to the anticipated frequency ν_0.

   (f) Find the decrement of the damped motion, i.e. the fraction by which the amplitude decreases during one period of motion.

4. [10 pts] A particle of mass m moves in one dimension under the influence of a force for which the potential energy is a periodic function of position x:

$$U(x) = U_0 \cos^2\left(\frac{\pi x}{2\ell}\right).$$



For a net energy $0 < E < U_0$, the particle is trapped in the vicinity of one of the minima of the potential energy and its motion is periodic in time.

(a) Relying on the second Newton's law, obtain an equation of motion for the particle.

(b) Consider the case of motion where the maximum displacement A of position from one of the minima is small compared to ℓ and expand the force to the lowest order in the displacement of the particle.

(c) Find an approximate period of motion τ_0 of the particle when the maximum displacement is small. Does that period depend on A?

(d) What is the maximal percentage error in the force when employing the expansion 4b, for $A = 0.1\ell$? What magnitude of percentage error do you correspondingly expect in the period τ_0 obtained using the expansion 4b, as compared to the real period τ for the particle? If the period τ for a given amplitude A were expanded in the powers of A/ℓ, what kind of powers would you expect in the expansion?

5. [5 pts]

(a) To solve the equation of a damped harmonic oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0.$$

assume the solution to be of the form

$$x(t) = f(t)\,e^{-\beta t},$$

and obtain an equation for $f(t)$.

(b) Obtain the solutions of the latter equation for $\omega_0 > \beta$ and $\omega_0 < \beta$. Show, further, that the general solution for $\omega_0 = \beta$ is $f(t) = A + B\,t$, where A and B are constants.

(c) Write out the corresponding forms of $x(t)$ for the three cases of β/ω_0.

6. [10 pts]

(a) Given $x(t) = A\,\mathrm{e}^{-\beta t}\cos(\omega_1 t - \delta)$ for an underdamped harmonic oscillator, with $\omega_1^2 = k/m - \beta^2$, obtain an expression for the time dependence of net energy, $E(t) = m\left(\dot{x}(t)\right)^2/2 + k\left(x(t)\right)^2/2$, with separated and organized smooth and oscillating contributions as a function of time. An example of the time dependence of the net energy is displayed in the top panel of Fig. 3-8 in the textbook.

(b) Obtain an expression for the rate of change of energy of the damped harmonic oscillator, again with separated and organized smooth and oscillating contributions, by directly differentiating the energy from 6a. An example of the rate of energy loss is displayed in the bottom panel of Fig. 3-8 in the textbook.

(c) Independently obtain the rate of energy change from the rate at which the resisting force carries out work, $\mathrm{d}E/\mathrm{d}t = \mathrm{d}W/\mathrm{d}t\big[\equiv P\big] = F_{\mathrm{res}}\,\dot{x} = -b\,\dot{x}^2 \leq 0$. Do the rates obtained in the two different ways agree?

(d) For a weakly damped harmonic oscillator with $\beta T_1 \ll 1$, find an approximate expression for the net energy averaged a period $\langle E\rangle(t) = (1/\tau_1)\int_t^{t+\tau_1} E(t')\,\mathrm{d}t'$.

(e) For a weakly damped harmonic oscillator with $\beta\tau_1 \ll 1$, find an approximate expression for the rate of energy loss averaged over a period $\langle P\rangle(t)$.

(f) Do the displacement amplitude and average energy decrease at the same rate? What is the difference if any?