PHY321 Homework Set 7

1. [10 pts] Consider critically damped harmonic oscillator, for which

$$x(t) = (A + B\,t)\,\mathrm{e}^{-\beta t}\,.$$

   (a) Express the constants A and B in terms of the initial values for the oscillator, $x(t = 0) = x_0$ and $\dot{x}(t = 0) = v_0$.

   (b) Sketch four phase-space trajectories for this oscillator, starting in the four different quadrants in the phase space corresponding to four combinations of signs of x_0 and v_0.

   (c) Prove analytically that a phase-space trajectory for the critically damped oscillator approaches the line $\dot{x} = -\beta\,x$ as $t \to \infty$.

   (d) Are the phase-space trajectories for the critically damped oscillator circling around the origin in phase space?

2. [10 pts] Consider a nondriven overdamped oscillator for which $x(t = 0) = x_0$ and $\dot{x}(t = 0) = v_0$.

   (a) Within the solution of the equation for the damped harmonic oscillator, written as

$$x(t) = A_1\,\mathrm{e}^{-(\beta+\omega_2)\,t} + A_2\,\mathrm{e}^{-(\beta-\omega_2)\,t}\,,$$

   where $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$, express the coefficients A_1 and B in terms of x_0 and v_0.

   (b) Consider a strongly overdamped oscillator where $\beta \gg \omega_0$. What is a qualitative difference in the t-dependence between the two terms in $x(t)$ with coefficients A_1 and A_2? Which term will dominate the changes in $x(t)$ at short times and which at long times and why?

   (c) Explain the qualitative features of the phase diagram for the overdamped oscillator in Fig. 3-11 of the textbook. Why are the early parts of the paths parallel to the line $\dot{x} = -(\beta + \omega_2)\,x$ and the late parts follow the line $\dot{x} = (\beta - \omega_2)\,x$?

3. [10 pts] Consider *velocity resonance curve* for a driven damped harmonic oscillator characterized by ω_0 and β parameters.

   (a) From $x(t) = A\cos(\omega t - \delta)/\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\,\beta^2}$, obtain an amplitude for velocity oscillations in the driven motion. For what frequency ω is the velocity amplitude maximal and what is the value of the maximal velocity?

   (b) The width of a resonance curve $\Delta\omega$ is defined as the difference between the two frequencies at the two sides of the resonance peak where the velocity amplitude drops to $1/\sqrt{2}$ of the maximal. Find the width of the velocity resonance curve.

   (c) Draw the velocity resonance curve for the case of an oscillator with $Q = 5$.

4. [5 pts] Consider an oscillator consisting of a 0.20 kg mass attached to a spring with a force constant of 5.0 N/m and immersed in a fluid that supplies a damping force represented by bv with $b = 0.30\,\text{kg/s}$.

   (a) What is the nature of the damping (over-, under- or critical)?

   (b) Next the oscillator is attached to an external driving force varying harmonically with time as $F_d(t) = F_0 \cos \omega t$, where $F_0 = 1.00\,\text{N}$ and $\omega = 4.0\,\text{rad/s}$. What is the amplitude of the resulting steady-state oscillations?

   (c) What is the resonant angular frequency of the system i.e. what is the frequency ω_R that the driving force has to be tuned to in order to produce a maximal amplitude?

   (d) What is that maximal amplitude?

   (e) What is the Q-value for this system?

5. [10 pts] Consider a driven harmonic oscillator for which the equation of motion is

$$m\,\ddot{x} = -k\,x - b\,\dot{x} + F_0\,\cos\omega t\,.$$

   (a) Given the stationary solution to the equation

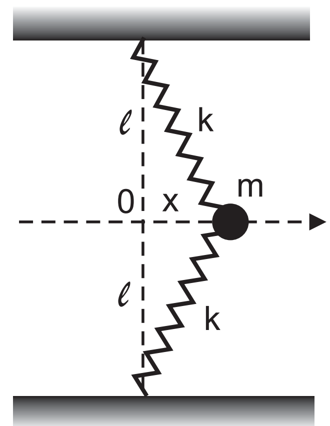
$$x(t) = D\cos\left(\omega t - \delta\right),$$

   where D and δ are functions of ω, find the instantaneous $P_{\text{drv}}(t) = F_{\text{drv}}\,\dot{x}$ and average power $\langle P_{\text{drv}} \rangle$ delivered by the driving force $F_{\text{drv}} = F_0\,\cos\omega t$. Simplify the results as much as possible, in particular attempting to separate the oscillating from constant terms in $P_{\text{res}}(t)$ and to illuminate the dependence on the phase shift δ.

   (b) Find the instantaneous $P_{\text{res}}(t) = F_{\text{res}}\,\dot{x}$ and average power $\langle P_{\text{res}} \rangle$ lost to the resistance force $F_{\text{res}} = -b\,\dot{x}$. Again, simplify the results as much as possible. Do the powers associated with the driving and resistance forces balance each other, whether instantaneously or on the average?

   (c) For what angular frequency ω is the average rate of energy transfer from driving to resistance force the maximal? What is the phase shift δ then?

6. [5 pts] Consider mass m attached by two identical springs, of spring constant k, to the rigid supports separated by distance of 2ℓ, as shown in the figure. The neutral length for each spring is $d < \ell$.



(a) What is the potential energy $U(x)$ for this system? Make a sketch of $U(x)$, assuming $d \sim \ell/2$. Make any characteristic features of $U(x)$ explicit.

(b) Expand the potential energy for small $x \ll \ell$, retaining one nonvanishing term beyond quadratic.

(c) What kind of nonlinearity is present in this system? How is such a system termed?

(d) Obtain a returning force from the expanded energy.

(e) Sketch phase diagrams for 3 energies of progressively increasing magnitude, illustrating transition from linear to nonlinear regime for this system.