PHY321 Homework Set 8

1. [5 pts] An object of mass m moves in one dimension under the influence of a resistance force either dependent (I) linearly on velocity, $F = -kmv$, or (II) cubically, $F = -kmv^3$.

   (a) From Newton's law, find the velocity $v(t)$ in the two cases, assuming that the object starts at a velocity v_0 at $t = 0$.

   (b) Demonstrate that in the case I, two solutions for $v(t)$, one for some starting v_{01} and another v_{02}, can be linearly combined to represent a valid solution to the equation of motion, representing some v_{03}. Can the solutions be combined in a similar manner for the case II of the force?

2. [10 pts] For an object of mass m moving vertically under the influence of the forces of gravity and air resistance

$$F = m\,\ddot{x} = -mg - km\dot{x}\,,$$

   we obtained in the past the velocity and position as a function of time (see Chapter 2 of the book):

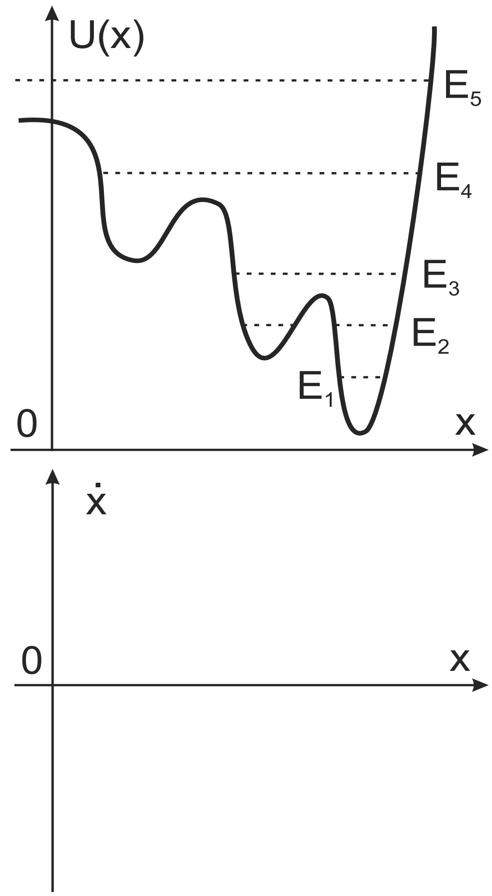
$$\begin{aligned} x &= h - \frac{g}{k}\,t + \frac{1}{k}\big(v_0 + \frac{g}{k}\big)\left(1 - \mathrm{e}^{-kt}\right), \\ \dot{x} &= -\frac{g}{k} + \big(v_0 + \frac{g}{k}\big)\,\mathrm{e}^{-kt}\,. \end{aligned}$$

   Here x-axis points in the vertical direction and h and v_0 are the position and velocity, respectively, at $t = 0$.

   (a) Consider an object with $k = 0.3\,\mathrm{s}^{-1}$ released from an elevation of $h = 100\,\mathrm{m}$ above the ground. By finding x and $\dot{x}$ for several t and by marking and joining the locations in the x-$\dot{x}$ plane, construct phase-space diagrams for the object released at three different velocities: $v_0 = 40\,\mathrm{m/s}$, 0 and $-40\,\mathrm{m/s}$. What form would those diagrams be reaching, if there were no ground and the object could fall down forever? In the terminology of phase-space diagrams, the set of phase-space locations that is approached in the course of an evolution, irrespectively of the initial conditions, is called a *strange attractor*.

   (b) Eliminate time t in favor of $\dot{x}$ and obtain a formal equation $x = x(\dot{x})$ for the phase diagram of the object.

3. [5 pts] A particle of mass m moves in one dimension under the influence of a conservative force for which the potential energy $U(x)$ is shown in the figure. Sketch phase diagrams for the 5 energies of the particle, indicated in the figure. Mark which phase diagram is for which energy.



4. [5 pts] A plane pendulum consisting of a mass m attached to a massless rod of length ℓ starts from rest at an angle θ_0. Given that the pendulum spends quarter of its period moving between angles $\theta = 0$ and $\theta = \theta_0$, the period can be calculated from

$$\frac{T}{4} = \int_0^{T/4} \mathrm{d}t = \int_0^{\theta_0} \frac{\mathrm{d}t}{\mathrm{d}\theta}\, \mathrm{d}\theta \,.$$

(a) Use energy conservation to obtain $\mathrm{d}\theta/\mathrm{d}t$ as a function of θ for a given θ_0. Use this result under the integral above to arrive at an integral expression for the period T as a function of the maximal angle θ_0. Comment on the presence or absence of the dependence of T on m, ℓ and θ_0.

(b) Use a small-angle expansion in the subintegral function, to arrive at the period in the limit $\theta_0 \to 0$. Does your result agree with that from solving the equation of motion in the small-angle approximation? Systematic corrections to the latter period, in powers of θ_0, may be arrived at by employing Taylor-expansions in the subintegral function of the expression obtained in 4a. Useful integral: $\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x$.

5. [10 pts] If an equation can be put into the form $x = f(x)$, one can attempt to solve that equation by successive substitution of the result from the l.h.s. to the r.h.s. The success of such a procedure depends on properties of the produced iterative map in the vicinity of the solution, and on the choice of a starting value for x. In the following, attempt to solve the listed nonlinear equations by rewriting them into an $x = f(x)$ form (not unique!) and by applying the method of successive substitution. Write out the consecutive values from your substitutions and comment on the outcome.

   (a) $x = \cos x$

   (b) $-1 + x^2\,(1 + \mathrm{e}^x) = 0$

   (c) $x^2 + x^5 = 1$

6. [5 pts] Consider the mapping function $x_{n+1} = \alpha \sin(\pi x_n)$. Here, α is a constant characteristic for the map, x is restricted to the range $0 < x < 1$ and the argument of the sine is in radians.

   (a) What is the asymptotic value of the mapping function when $\alpha = 0.6$? Depending on your starting value, the asymptotic value should be reached, to a very good approximation, in ~ 20 iterations.

   (b) What are the *two* asymptotic values for the mapping function when $\alpha = 0.73$?