PHY321 Practice Midterm 3

Contextual information:

$G = 6.673 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$

Calculus in spherical coordinates:

$\vec{\mathrm{grad}}\,\Psi = \vec{\mathrm{e}}_r \frac{\partial \Psi}{\partial r} + \vec{\mathrm{e}}_\theta \frac{1}{r} \frac{\partial \Psi}{\partial \theta} + \vec{\mathrm{e}}_\phi \frac{1}{r \sin\theta} \frac{\partial \Psi}{\partial \phi}$

$\vec{\mathrm{div}}\,\vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 A_r\right) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta} \frac{\partial A_\phi}{\partial \phi}$

$\nabla^2 \Psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Psi}{\partial \phi^2}$

1. At time $t = 0$, a student releases a small ball of mass m from an elevation h above the ground. The ball drops to the floor, collides elastically with the floor and rises back to its original position where it is recaptured by the student.

   (a) [2 pts] With x representing the ball's elevation above the ground, derive an equation for the phase diagram of the ball, when the ball is in flight.

   (b) [3 pts] Draw the phase diagram of the ball for the time between the release and recapture of the ball. Indicate the direction for the diagram. Mark values on the axes where the diagram meets the axes. Derive the values if necessary.

2. [3 pts] Determine the asymptotic behavior for the logistic map:

$$x_{\mathrm{n+1}} = 0.01\, x_n\, (1 - x_n)\,,$$

where $0 < x < 1$.

(Over)

3. Astronauts are visiting the small asteroid Zarg, and want to know what is inside it. They drill a hole all the way through the asteroid, and then drop in a small probe equipped with an accelerometer. The probe oscillates back and forth between the starting point and the other side of Zarg, and sends acceleration data back to the astronauts. The data indicate that the acceleration of the probe can be written as $\vec{a} = -C\, r^2\, \hat{r}$ where C is a constant and r is the distance from the center of Zarg.

   (a) [6 pts] Assuming that Zarg is spherically symmetric, use Gauss' law to find its density as a function of distance from the center, $\rho(r)$.

   (b) [3 pts] Use the Shell Theorem to check your answer. (This means, use your answer for $\rho(r)$ to calculate the acceleration of the probe as a function of r.)

(Over)

4. A satellite is in a circular orbit of radius R around Earth of mass $M = 5.97 \times 10^{24}$ kg.

   (a) [3 pts] How is the velocity v of the satellite related to the radius R, mass M and gravitational constant G?

   (b) [5 pts] What needs to be the radius R to make the orbit semisynchronous, i.e. with a period of 12h? (GPS satellites move on such orbits.) Obtain a value for R.