

Newtonian Dynamics

In the next few weeks, we'll study two chapters from the textbook,

Thornton and Marion, Classical Dynamics (5th ed.)

The chapters are

Chapter 2: Newtonian Dynamics for a Single Particle

Chapter 9: Dynamics for a System of Particles

Classical dynamics is based on Newton's three laws of motion ...

1. An object in motion has constant velocity if the net force acting on the object is 0.
2. If the net force acting on an object is F , then the acceleration a is $a = F/m$.
3. For every action (F) there is an equal but opposite reaction ($-F$).

You have already studied some implications in the notes from Prof. Stump. Now we will learn some mathematical techniques to apply the *laws of motion*.

Chapter 2 is concerned with Newton's second law,

$$\vec{a} = \frac{\vec{F}}{m}, \quad \text{or} \quad \vec{F} = m \vec{a}$$

Here m = mass of a particle; \mathbf{F} = the force acting on the particle; and \mathbf{a} = the acceleration of the particle. Force and acceleration are *vectors*, so in handwritten equations \mathbf{F} and \mathbf{a} should be written with an arrow over the symbol. (For one dimensional motion, the arrow is usually omitted.)

Newton's second law is a differential equation,

$$\frac{d\vec{v}}{dt} = \frac{\vec{F}}{m}$$

so in Chapter 2 we will be interested in methods to solve differential equations.

Kinematics and Dynamics

"Kinematics" is the description of motion.

The important quantities are position \vec{x} , velocity \vec{v} and acceleration \vec{a} .

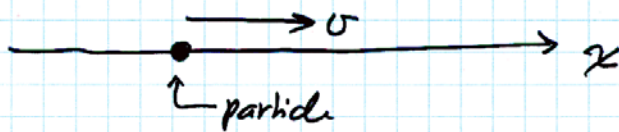
"Dynamics" is the relationship between force and motion. According to Newton's

second law, $\vec{a} = \vec{F}/m$.

Kinematics and Dynamics

Consider the three elementary examples from Lectures 1a, 1b and 1c.

- One-dimensional motion



Kinematics:

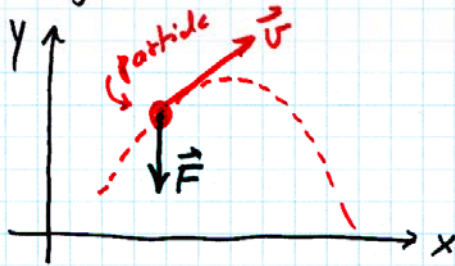
$$\vec{v} = \frac{d\vec{x}}{dt}; \quad \vec{a} = \frac{d\vec{v}}{dt}$$

Dynamics:

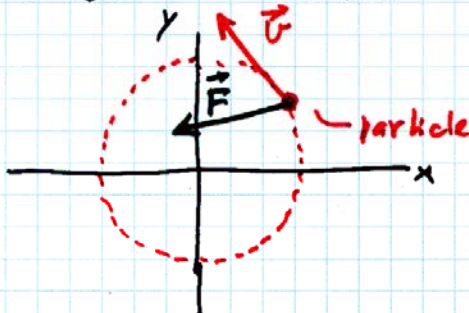
$$\vec{a} = \vec{F}/m$$

If $v > 0$ and $F > 0$, speed increases;
if $v > 0$ and $F < 0$, speed decreases.

- Projectile Motion



- Circular Motion



Chapter 9 is concerned with Newton's third law. In a system of particles,

$$\sum_{\text{all forces}} \mathbf{F}_i = \sum_{\text{external forces}} \mathbf{F}_i^{\text{ext}} ;$$

or, equivalently,

$$\sum_{\text{internal forces}} \mathbf{F}_i^{\text{int}} = 0.$$

These equations are consequences of Newton's third law. For example, consider a system of just 2 particles:

$$\mathbf{F}_1 = \mathbf{F}_1^{\text{ext}} + \mathbf{f} \quad \text{and} \quad \mathbf{F}_2 = \mathbf{F}_2^{\text{ext}} - \mathbf{f}$$

where \mathbf{f} = the force on m_1 due to m_2 ; by Newton's third law, the force on m_2 due to m_1 must be $-\mathbf{f}$.

$+\mathbf{f}$ and $-\mathbf{f}$ are the internal forces, i.e., internal to the system. They cancel if we calculate the sum of forces.

Reading and problem assignments:

❖ Read Chapter 2 from the textbook, Thornton and Marion, ***Classical Dynamics***.

❖ Do the LON-CAPA problems entitled "Homework Set 2a".

Dynamics for a single particle (CHAPTER 2)

Our goal is to solve differential equations with this form (Newton's second law)

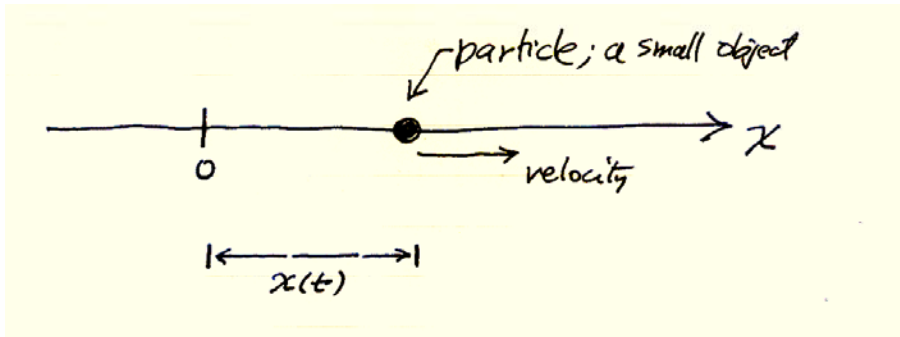
$$m \frac{d\vec{v}}{dt} = \vec{F} \quad \text{with} \quad \frac{d\vec{x}}{dt} = \vec{v}$$

m = mass [kg]
 v = velocity [m/s]
 F = force [N]
 x = position [m]

Note that \mathbf{x} , \mathbf{v} , *and* \mathbf{F} are vectors. In handwritten equations they should be written with an arrow over the symbol. (For one-dimensional motion the arrow is usually omitted.) Also, physical quantities have **units of measurement** [kg, m, s, etc] which must be given in any numerical calculations.

In general the force \mathbf{F} may depend on position \mathbf{x} , velocity \mathbf{v} , and time t . That is, \mathbf{F} would be a function of all three variables $\mathbf{F} = \mathbf{F}(\mathbf{x}, \mathbf{v}, t)$. However, in most examples \mathbf{F} is independent of some of the variables.

Three cases for one-dimensional motion



The equations of motion are $m \, dv/dt = F$ and $dx/dt = v$.

Case 1. $F = F(t)$

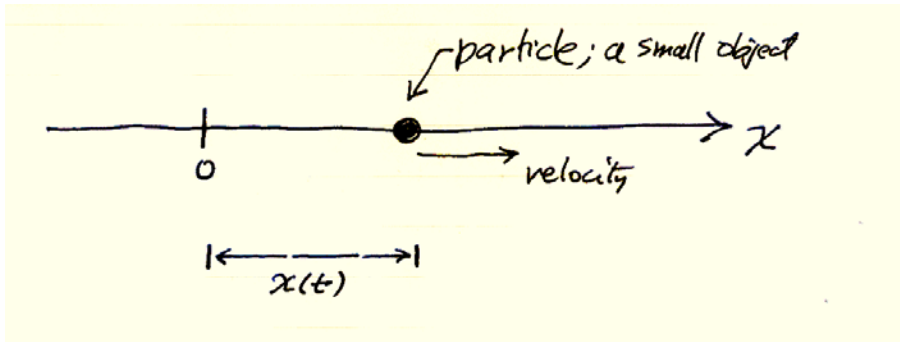
In this case the equation of motion is solved immediately by integration

$$\frac{dv}{dt} = \frac{F(t)}{m} \quad \text{implies} \quad v(t) - v_0 = \int_{t_0}^t \frac{F(t')}{m} dt'$$

This is an example of the *fundamental theorem of calculus*.

THE FUNDAMENTAL THEOREM OF CALCULUS

$$\int_{x_1}^{x_2} \frac{df}{dx'} dx' = [f]_{x_1}^{x_2} = f(x_2) - f(x_1)$$



The equations of motion are $m \, dv/dt = F$ and $dx/dt = v$.

Case 2. $F = F(v)$

In this case the equation of motion is solved by *separation of variables*

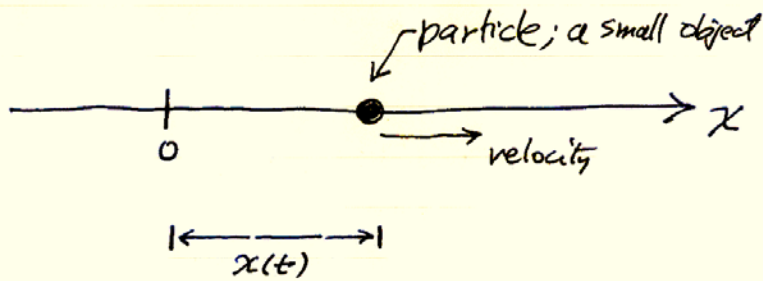
$$\frac{dv}{dt} = \frac{F(v)}{m} \quad \text{implies} \quad \frac{dv}{F(v)} = \frac{dt}{m}$$

integrate both sides of the equation

$$\int_{v_0}^v \frac{dv'}{F(v')} = \int_{t_0}^t \frac{dt'}{m} = \frac{t - t_0}{m}$$

Now, you have to evaluate the integral $dv/F(v)$. But after that is done, the equation relates v to t . In principle we then know $v(t)$. The position is obtained by integrating the velocity

$$x(t) - x_0 = \int_{t_0}^t v(t') \, dt'$$



The equations of motion are $m \, dv/dt = F$ and $dx/dt = v$.

Case 3. $F = F(x)$

In this case the equation of motion is solved by *the method of potential energy*.

If F is a conservative force then we can write $F(x) = -dU/dx$ where $U(x) =$ potential energy, a function of position.

$$\frac{dv}{dt} = \frac{F(x)}{m} \quad \text{implies} \quad mv \frac{dv}{dt} = v F = - \frac{dx}{dt} \frac{dU}{dx}$$

Note then that $\frac{d}{dt} \left(\frac{1}{2} mv^2 \right) = - \frac{dU}{dt}$

so $\frac{1}{2} mv^2 + U =$ a constant $= E$, the energy.

OK, now write $\frac{dx}{dt} = \pm \sqrt{\frac{2}{m} [E - U(x)]}$

and solve *this* equation by separation of variables

$$\int_{x_0}^x \frac{dx'}{\sqrt{E - U(x')}} = \pm \sqrt{\frac{2}{m}} (t - t_0)$$

After you do the integral, you will know the relationship between t and x , depending on the energy E .

Exercise 1. Suppose the force F is constant. Then any of the three methods could be used to determine the motion.

Show how to determine the position $x(t)$ as a function of time using each method. You must get the same answer by all three methods!

Exercise 2. A mass m moves in one dimension x subject to a frictional force $F = -\gamma v$.

(a) Determine $x(t)$ if the initial position is 0 and the initial velocity is v_0 . (b) The particle slows and eventually comes to rest. Determine the final position.

Reading and problem assignments:

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❖ Do the LON-CAPA problems entitled "Homework Set 2a".

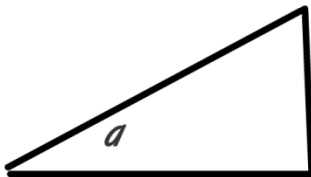
Dynamics of a single particle

Examples with Friction

In this lecture we'll consider some simple examples, which we'll analyze using vectors and calculus.

ONE. A block on an inclined plane (part 1).


A block is placed on an inclined plane, which has angle of inclination θ with respect to the horizontal. Determine the maximum angle θ such that the block will stay put.



The maximum angle is $\arctan \mu_s$.

TWO. A block on an inclined plane (part 2).

A block is placed on an inclined plane, which has angle of inclination θ with respect to the horizontal. Now suppose the angle is large enough such that the block slides down the plane. Determine the acceleration.



The diagram shows a block on an inclined plane at angle θ . The left part shows the block with an acceleration vector \vec{a} pointing down the slope. The right part is a free-body diagram showing forces: normal force N perpendicular to the slope, friction force F_f up the slope, and weight mg acting vertically downwards. The weight is decomposed into components $mg \sin \theta$ down the slope and $mg \cos \theta$ perpendicular to the slope.

Equations of motion: $N = mg \cos \theta$
($m\vec{a} = \vec{F}$) $ma = mg \sin \theta - F_f$

Coefficient of friction: $F_f = \mu_k N$

$\therefore a = g \sin \theta - \mu_k g \cos \theta$

$$a = g (\sin \theta - \mu_k \cos \theta)$$

THREE. A linear retarding force (part 1) (Example 2.4)

A particle moves in one dimension in a medium with a linear retarding force, $F_R = -\gamma v$. If the initial velocity is v_0 , how far will the particle move before it comes to rest?



The final distance is $m v_0/\gamma$.

FOUR. A linear retarding force (part 2) (Example 2.5)

A particle moves in one dimension in a medium with a linear retarding force, $F_R = -\gamma v$. Also, it is pulled by a constant force F_0 . Assume the initial velocity is 0. (A) Determine the velocity as a function of time. (B) Sketch of graph of $v(t)$, and describe the motion in detail.

(A) The equation of motion is $m \frac{dv}{dt} = F_0 - \gamma v$.

- Solution Method #1: Separation of variables (exercise)
- Solution Method #2: Linear inhomogeneous equation

$v = \text{particular solution} + \text{solution of the homogeneous equation}$

$$v = \frac{F_0}{\gamma} + C e^{-\gamma t/m}$$

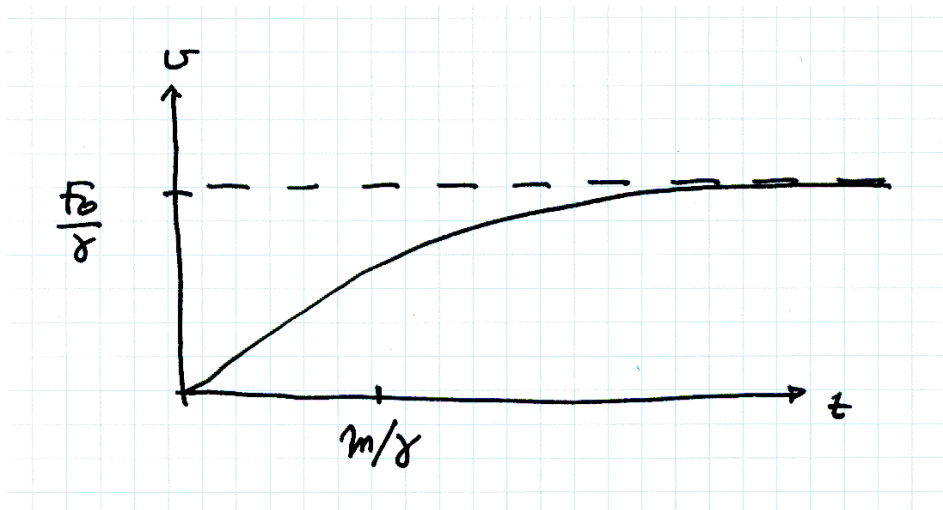
The initial condition is $v(0) = 0$, which determines $C = -F_0/\gamma$.

$$v(t) = \frac{F_0}{\gamma} \left\{ 1 - e^{-\gamma t/m} \right\}$$

FOUR, part B

$$v(t) = \frac{F_0}{\gamma} \left[1 - \exp(-\gamma t / m) \right]$$

First sketch a graph of the function $v(t)$.



- The initial velocity is 0.
- For small times, we may approximate the velocity by a Taylor series approximation, and the result is

$$v(t) \approx \frac{F_0}{m} t \quad \text{for} \quad \gamma t \ll m$$

... the same as if only the constant force were acting.

- In the limit $t \rightarrow \infty$, the velocity approaches a constant, F_0/γ . This is the **terminal velocity**.

$$v(t) \approx \frac{F_0}{\gamma} \quad \text{for} \quad \gamma t \gg m$$

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Dynamics of a Single Particle
Conservation Laws

A conservation law is a statement that some quantity is constant, i.e., conserved.

A conservation law is important because it places a strict limitation on the changes that can occur in a physical system.

In mechanics, three conservation laws are often encountered:

- conservation of momentum*
- conservation of angular momentum*
- conservation of energy*

In this mini-lecture we'll consider each of these, for the dynamics of a single particle.

Conservation of Momentum

In the dynamics of a single particle, the momentum (p) is defined by $p = m v$.

Theorem $\frac{d\mathbf{p}}{dt} = \mathbf{F}$

Proof. $\frac{d\mathbf{p}}{dt} = m \frac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{F},$

by Newton's second law.

So Newton's second law for a single particle can be written as $dp/dt = \mathbf{F}$.

Corollary. For an isolated particle, p is conserved.

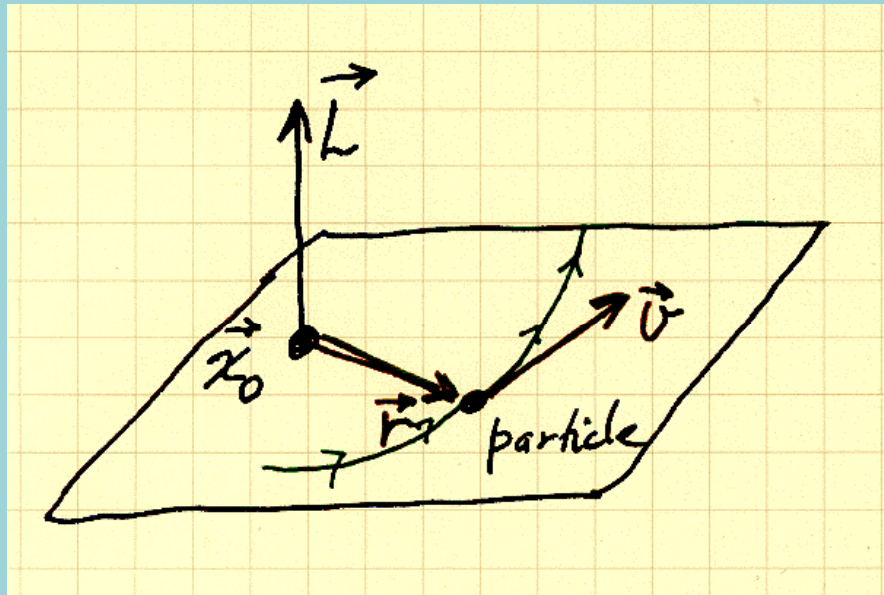
Proof. For an isolated particle, the force on the particle is 0; $\mathbf{F} = 0$. Thus $dp/dt = 0$; or, p is constant.

QED

Comment. In the dynamics of a single particle, conservation of momentum is not a very powerful concept. It applies only to an isolated, free particle. [It's just the same as Newton's first law of motion --- \mathbf{v} is constant if $\mathbf{F} = 0$.] But in the dynamics of a system of particles, conservation of total momentum is a very powerful concept. (Chapter 9).

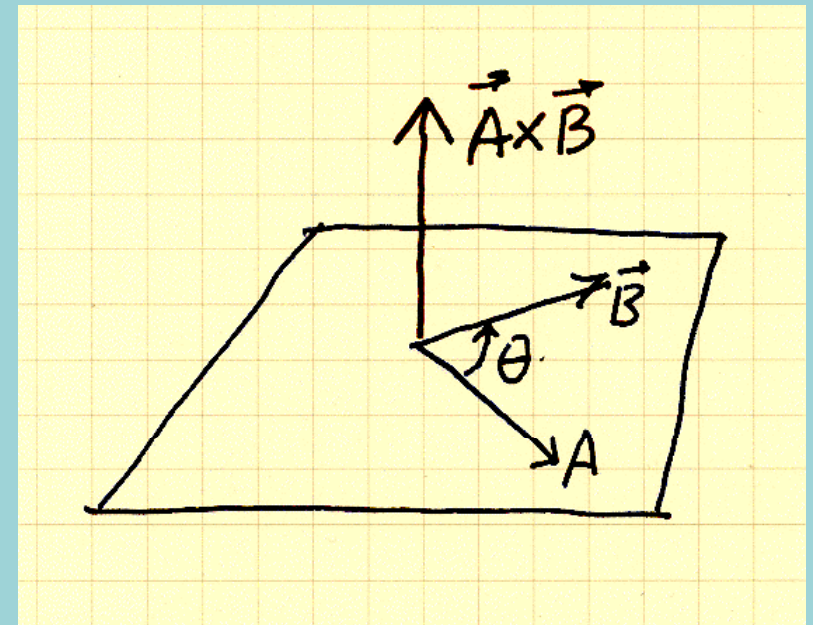
Angular Momentum

In the dynamics of a single particle, the angular momentum (L) around a fixed point x_0 is defined by $L = r \times p$ where $r = x - x_0$.



[x is the position of the particle; r is the vector from x_0 to x . If the fixed point is the origin, $x_0 = 0$, then $r = x$.]

Cross product of vectors



[If two vectors \mathbf{A} and \mathbf{B} cross, then they define a plane. The cross product of \mathbf{A} and \mathbf{B} , denoted $\mathbf{A} \times \mathbf{B}$, is a third vector, perpendicular to the plane spanned by \mathbf{A} and \mathbf{B} . The direction of $\mathbf{A} \times \mathbf{B}$ is given by the right hand rule; the magnitude is $A B \sin \theta$. See Lecture 1d.]

Conservation of Angular Momentum

In the dynamics of a single particle, the angular momentum (L) around a fixed point x_0 is defined by $L = r \times p$ where $r = x - x_0$.

Theorem.

$$\frac{dL}{dt} = r \times F$$

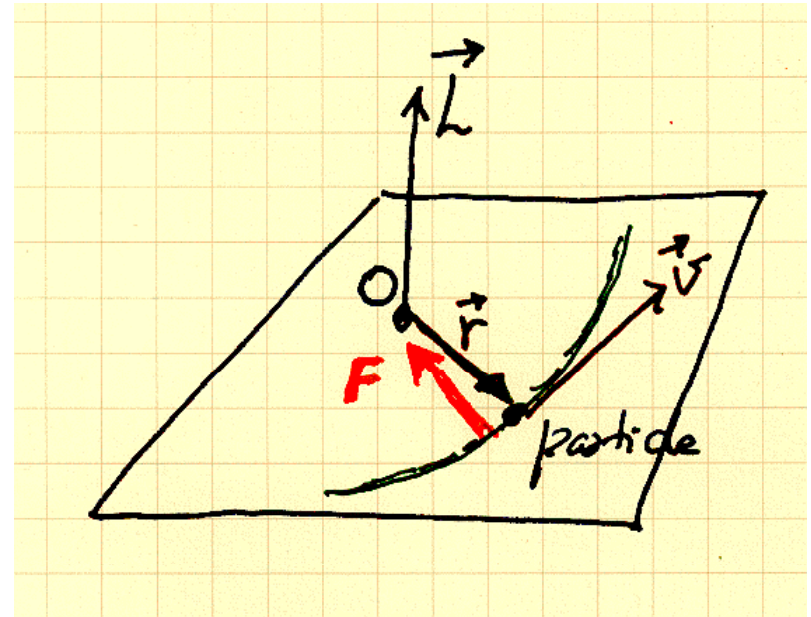
Note: $r \times F$ = the torque about x_0 .

Proof.

$$\begin{aligned} \frac{dL}{dt} &= \frac{d}{dt}(r \times p) \\ &= \frac{dr}{dt} \times p + r \times \frac{dp}{dt} \\ &= v \times p + r \times F \end{aligned}$$

But $v \times p = m v \times v = 0$. (*)

QED



Corollary. For a central force, L is conserved.

Proof. A "central force" means a force on m directed toward or away from the fixed point x_0 . For a central force, F and r are parallel (or anti-parallel) so $r \times F = 0$.

Then $dL/dt = 0$; or, L is constant.

Comment. Conservation of angular momentum will be very important in orbit calculations (Chapter 7).

(*) The cross product of parallel vectors is always 0.

Conservation of Energy

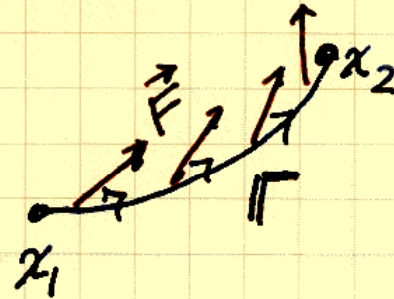
Energy is a little more subtle than momentum, requiring some preliminary definitions.

$$\text{Work} = \int_{\Gamma} \vec{F} \cdot d\vec{x}$$

$$\text{Kinetic Energy} = \frac{1}{2} m v^2$$

$$\text{Potential Energy } U(\vec{x})$$

$$\Delta U = - \int_{\Gamma} \vec{F} \cdot d\vec{x} ; \text{ or, } \vec{F} = -\nabla U$$
$$\{ U(\vec{x}_2) - U(\vec{x}_1) \}$$



Recapitulation. For a conservative force, we write $\mathbf{F}(\mathbf{x}) = -\nabla U$ where $U(\mathbf{x})$ is the potential energy.

Theorem 1

(Work - Kinetic Energy)

$$\Delta K = W$$

Proof. $\Delta K = \int \frac{dK}{dt} dt$

$\Delta K = \int m \vec{v} \cdot \frac{d\vec{v}}{dt} dt$ $\leftarrow K = \frac{1}{2} m v^2$

$= \int \left(m \frac{d\vec{v}}{dt} \right) \cdot (\vec{v} dt) = \int \vec{F} \cdot d\vec{x} = W$

Theorem 2

(Conservation of Energy)

In the dynamics of a single particle moving under the influence of a conservative force F , $K + U$ is a constant of the motion. That is, the total energy is conserved.

Proof. $\frac{dE}{dt} = \frac{dK}{dt} + \frac{dU}{dt} = m \vec{v} \cdot \frac{d\vec{v}}{dt} + \nabla U \cdot \frac{d\vec{x}}{dt}$

$= \vec{v} \cdot \left[m \frac{d\vec{v}}{dt} - \vec{F} \right] = 0$

by Newton's 2nd law

$\therefore E$ is constant.

The next lecture will be some examples in which conservation of energy is used to analyze the dynamics of a particle.

