Math—19 Jan 2012

- Announcements
  - Homework 1 is due on Tues, the 24th.
  - Office hours: Tues, Thurs, 12:00–1:00
  - A few scribes have not sent answers to their questions. Get them in!
- Outline
  - 4-vectors
  - Spacetime diagram, t and x axis in a moving frame
  - Transformations
  - More sophisticated notation
  - What is conserved?

4-vectors

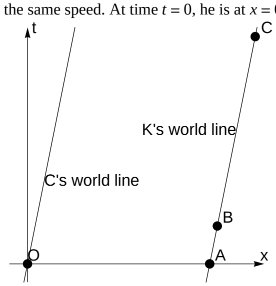
The position of an event (t, x, y, z) is a 4 vector.

Q: In what ways is it different from a set of 4 numbers?

Spacetime diagram

A spacetime diagram is a plot of position and time. We use the same units for space and time; e. g., meters and time-meter (the time for light to travel a meter).

Kristen is moving at speed 0.2. At time $t = 0$, she is at $x = 1$. Her worldline (position vs time) is on the plot. Carlos is moving at the same speed. At time $t = 0$, he is at $x = 0$.



Consider the distance^2 between the origin and events on her worldline. Calculate it with the Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

In[622]:=
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dist2[t_, v_] := (1 + v t)^2 - t^2;
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In[625]:=
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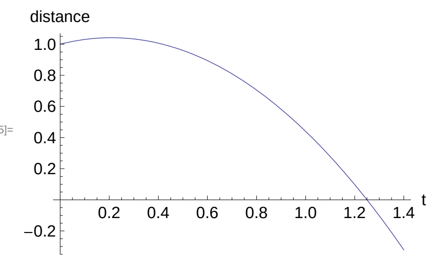
Plot[dist2[t, .2], {t, 0, 1.4}, ImageSize → 300,
 Evaluate@bs, AxesLabel → {"t", "distance"}]
```

Out[625]=



Q: The distance^2 = 0 for some event. Why? What is the relationship between x and t for this event?

- **What event happens at the same time as event O in Kristen's frame?**

Why is the distance is a maximum for some time?

To find the time for which distance is a maximum, $\frac{d}{dt}\left[(1 - v\,t)^2 - t^2\right] = 0$. The distance is a maximum at

$$(t, x) = \left(v/(1 - v^2),\ 1/(1 - v^2)\right).$$

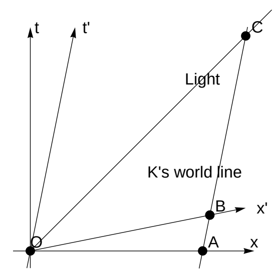
The maximum distance is $1/(1 - v^2)$.

Recall that distance is an invariant. its is the same in my frame of reference and in Kristen-Carlos' frame.

Because the events on Kristen's world line are at the same x' in Kristen-Carlos' frame,

$$s^2 = -t'^2 + x'^2$$

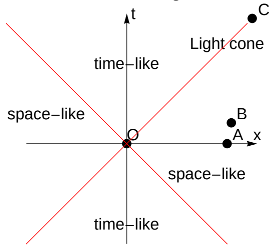
is a maximum for $t' = 0$. Events O and B occur at the same time in Kristen-Carlos frame.
We have figured out the x' axis.



- **Regions in spacetime**

We have figured out events for which the distance^2 is greater than 0 and 0. Since distance is invarient, these events map out regions that have a physical meaning.

$s^2 > 0$ for events O and A. Event A is in a region with space-like separation.
$s^2 = 0$ for events O and C. Event O is on the light cone.
$s^2 < 0$ Time-like region



- **Plots**

Transformations

- **Lorentz transformation**

We found the event $(t, x) = \left(v/(1-v^2), 1/(1-v^2)\right)$ transforms to $(t', x') = \left(0, 1/(1-v^2)\right)$ in Kristen-Carlos' frame.

The coordinates of an event (t, x, y, z) become in a frame moving at v in the x direction

$$t' = (t - v\,x)\Big/\left(1-v^2\right)^{1/2}$$

$$x' = (x - v\,t)\Big/\left(1-v^2\right)^{1/2}$$

$$y' = y$$

$$z' = z$$

- **Rotation**
- **Translation**
- **Scale**

Operations that make new 4-vectors

A 4-vector is an object that has the same transformations as (t, x, y, z).

A scalar is an object that is invariant under transformations.

All of the transformations are linear. Therefore if A and B are 4-vectors and a and b are scalars, $a\,A + b\,B$ is a 4 vector.

4-velocity

Construct a new 4-vector from $(d\,t, d\,x, d\,y, d\,z)$.

$d\,\tau = \left[d\,t^2 - \left(d\,x^2 + d\,y^2 + d\,z^2\right)\right]^{1/2}$ is a scalar.

Define the 4-velocity

$$u \equiv \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$$

Since this is the product of a 4-vector and a scalar, it is a 4-vector.

The components are

$$u = \left(\gamma, \gamma\,v_x, \gamma\,v_y, \gamma\,v_z,\right),$$

where $\gamma = \left(1 - v^2\right)^{-1/2}$.

The length^2 of u is −1.

4-momentum

Construct a new 4-vector from the 4-velocity. Multiply by the rest mass, the mass observed in a frame in which the particle is at rest. The 4-momentum is

$$p \equiv m\,u.$$

In components,

$$p = m\,\gamma\left(1, v_x, v_y, v_z\right)$$

or

$$p = \left(E, p_x, p_y, p_z\right)$$

Q: What is the length of p?
Q: I already know the transformation of p to a frame that is moving with respect to me. Why?

More sophisticated notation

- **4 - vectors written as contravarient vectors**

Write a 4-vector a as its components in this way:

$$\left(a^0, a^1, a^2, a^3\right)$$

The 0-th component is the time component. The 1, 2, 3 components are x, y, z.
Or this way:

$$a^\mu$$

Use a Greek letter as a superscript. This is called a contravarient 4 vector.

- **Metric tensor**

The metric is written $g_{\mu\nu}$. Letter g is used.
This is a 4×4 tensor. (We will learn what a tensor is later.) The Schwartzschild metrics that we encountered are diagonal.

Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Schwartzschild metric

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-2\,\frac{M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-2\,\frac{M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

- **Contraction**

The length of a contravariant vector is

$$a\cdot a \equiv \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} a^\mu a^\nu g_{\mu\nu}$$

Rule: If the same Greek letter occurs in both the subscript and superscript, sum over it. With is rule,

$$a\cdot a \equiv a^\mu a^\nu g_{\mu\nu}$$

- **Covariant vectors**

A covarient vector is the contraction of a contravariant vector and the metric tensor.

$$a_\mu = a^\nu g_{\mu\nu}$$

Example: A contravariant 3-vector is

$$a^\mu = (dr,\, d\theta,\, d\phi)$$

The metric tensor is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The covariant vector is

$$a_\mu = \left(dr,\, r^2\, d\theta,\, r^2\sin^2\theta\, d\phi\right)$$

Q: If I measured the distance between two points that differ in θ only, would I get the contravariant component, or the covariant component, or something else?

Spacetime near a star

The contravariant vector between two events is

$$dx = (dt,\, dr,\, d\theta,\, d\phi).$$

The metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-2\,\frac{M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-2\,\frac{M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The contravariant 4-momentum is

$$p^\mu = m\left(\frac{dt}{d\tau},\, \frac{dr}{d\tau},\, \frac{d\theta}{d\tau},\, \frac{d\phi}{d\tau}\right).$$

The covariant 4-momentum is

$$p_\mu = m\left(-\left(1-2\,\frac{M}{r}\right)\frac{dt}{d\tau},\, \left(1-2\,\frac{M}{r}\right)^{-1}\frac{dr}{d\tau},\, r^2\,\frac{d\theta}{d\tau},\, r^2\sin^2\theta\,\frac{d\phi}{d\tau}\right).$$

Q: If I measure the momentum in the ϕ direction, what do I get? What is it called?

Conservation laws, an example

Gravitational redshift reconsidered

We found ..

For $r_1 = \infty$,

$$\nu(r) = \nu_\infty\left(1-2\,\frac{M}{r}\right)^{-1/2}. \qquad (1)$$

Since energy $E = h\,\nu$, ν is directly proportional to energy. We know how

The contravariant 4-momentum is

$$p^\mu = \left(p^0,\, p^r,\, p^\theta,\, p^\phi\right).$$

The covariant 4-momentum in the time direction

$$p_0 = g_{0\,\mu}\, p^\mu$$

Since the metric is diagonal,

$$p_0 = g_{00}\, p^0.$$

The measured energy is E_m.

$$E_m^2 = -\left(p^0\right)^2 g_{00}$$

Rewrite as

$$E_m^2 = -\left(p^0\right)^2 g_{00} = -\left(g_{00}\, p^0\right)^2 g_{00}^{-1} = -(p_0)^2\, g_{00}^{-1}.$$

Substiture Equation 1 to get

$$(h\,\nu_\infty)^2\, g_{00}^{-1} = -(p_0)^2\, g_{00}^{-1}$$

Therefore

$$\boxed{p_0 = h\,\nu_\infty}$$

Noether's theorem

We have discovered a case of Noether's theorem. If the metric does not change by translating a coordinate γ, then the conjugate momentum p_γ is conserved.

Here the metric does not depend on time. Therefore Noether's theorem say p_0 is conserved.

Q: Apply Noether's theorem to the x-coordinate in flat space with coordinates $(t,\, x,\, y,\, z)$. What is conserved?