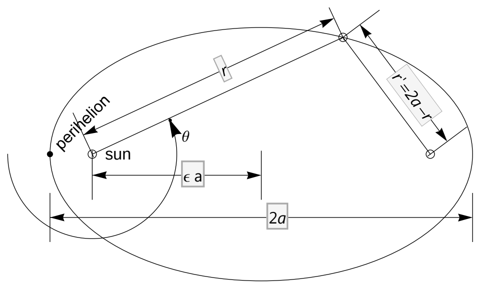
Precession of Mercury’s orbit—24 Jan

- Announcements
  - Homework 2 is due on 1/31. The link is on the syllabus on angel.
- Outline
  - Introduction
  - Conserved quantities
  - Equation for the orbit
  - Calculation of precession
  - Temporal order

Precession of the perihelion of Mercury

In Newton’s theory of gravity, the orbit of a planet around a single star is an ellipse. In Einstein’s theory, the orbit is not closed.



Some definitions:
The equation for an ellipse is

$$r(\phi) = a\left(1-\epsilon^2\right)/[1-\epsilon\cos(\phi-\phi_0)]$$

where r is the distance to the sun
a is the semimajor axis
ϵ is the eccentricity
r is the distance to the sun
θ is the true anomaly.

In one century (500 orbits), the perihelion of Mercury advances by 43arcsec. There are much larger effects (10 times larger) due to the gravity of other planets.

The nonzero perihelion shift of Mercury was a longstanding problem. Einstein calculated it from his theory in 1915. Einstein, A., 1915, Sitz., Preuss. Akad. Wis., 831.

“explains... qualitatively... the secular rotation of the orbit of Mercury, discovered by Le Verrier, ... without the need for and special hypothesis.”—Einstein, quotated in Pais

"For a few days, I was beside myself with joyous excitement."—Einstein, quoted in Pais.

Outline of the calculation

1. The metric determines the conserved quantities, the angular momentum and energy.
2. Derive the equation for the distance vs true anomaly $r(\phi)$.
3. Find the deviation from Newton’s solution.

Plots

Conserved quantities

Noether’s theorem

Noether’s theorem: If the metric does not change by translating a coordinate γ, then the conjugate momentum p_γ is conserved. (We will prove it in March when discussing equation of motion.)

http://owpdb.mfo.de/detail?photo_id=9267
Amalie Noether, March 23, 1882 - April 14, 1935.

Conserved quantities

The contravariant vector between two events is

$$dx = (dt,\, dr,\, d\theta,\, d\phi).$$

The metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-2\frac{M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-2\frac{M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Here the metric does not depend on time. Therefore Noether’s theorem say p_0 is conserved.
Here the metric does not depend on ϕ. Therefore Noether’s theorem say p_ϕ is conserved.

The contravariant 4-momentum is

$$p^\mu = m\left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\phi}{d\tau}\right).$$

The covariant 4-momentum is

$$p_\mu = m\left(-\left(1 - 2\,\frac{M}{r}\right)\frac{dt}{d\tau},\ \left(1 - 2\,\frac{M}{r}\right)^{-1}\frac{dr}{d\tau},\ r^2\,\frac{d\theta}{d\tau},\ r^2\sin^2\theta\,\frac{d\phi}{d\tau}\right).$$

We know from Kepler that the orbit is in a plane. Choose $\theta = \pi/2$.

Define E and L to be the conserved energy and angular momentum.

$$E = m\left(1 - 2\,\frac{M}{r}\right)\frac{dt}{d\tau}$$

$$L = m\,r^2\,\frac{d\phi}{d\tau}$$

Define the energy and angular momentun per unit mass.

$$e = \left(1 - 2\,\frac{M}{r}\right)\frac{dt}{d\tau}$$

$$l = r^2\,\frac{d\phi}{d\tau}$$

- **Interpret the conserved energy**

What is $\frac{dt}{d\tau}$? On Thurs we found

$$\frac{dt}{d\tau} = \left(1 - v^2\right)^{-1/2}$$

For small v,

$$\frac{dt}{d\tau} \approx 1 + \frac{1}{2}\,v^2 \quad \text{(a)}$$

Q: What is the conserved energy e in the absence of gravity?

Q: Is the correction for gravity comparable to or much smaller than the 2nd term in equation (a)?

Equation of the orbit

Momenta

$$p_\mu/m = \left[-e,\ \left(1 - 2\,\frac{M}{r}\right)^{-1}\frac{dr}{d\tau},\ 0,\ l\right]$$

$$p^\mu/m = \left[\left(1 - 2\,\frac{M}{r}\right)^{-1} e,\ \frac{dr}{d\tau},\ 0,\ \frac{l}{r^2}\right]$$

Since $p^2 = -m^2$,

$$-\left(1 - 2\,\frac{M}{r}\right)^{-1} e^2 + \left(1 - 2\,\frac{M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{l^2}{r^2} = -1.$$

Rearrange. Define an effective potential

$$\boxed{V_{\text{eff}}(r) = -\frac{M}{r} + \frac{l^2}{2\,r^2} - \frac{M\,l^2}{r^3}}$$

Then

$$\boxed{\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{d\,r}{d\,\tau}\right)^2 + V_{\text{eff}}(r)}$$

Q: Interpret the terms $\frac{e^2-1}{2}$, $\frac{1}{2}\left(\frac{d\,r}{d\,\tau}\right)^2$, $-\frac{M}{r}$, $\frac{l^2}{2\,r^2}$

The term $-\frac{M\,l^2}{r^3}$ is new. It is not present with Newton's theory of gravity.

Newtonian values

Specific angular momentun (angular momentum per unit mass)

$$l^2 = M\ a\left(1 - \epsilon^2\right)$$

$$\frac{1}{2}\left(e^2 - 1\right) = -\frac{1}{2}\,M/a$$

$$r(\phi) = \left\{\frac{M}{l^2}\left[1 - \epsilon\cos(\phi - \phi_0)\right]\right\}^{-1} = a\left(1 - \epsilon^2\right)\Big/\left[1 - \epsilon\cos(\phi - \phi_0)\right]$$

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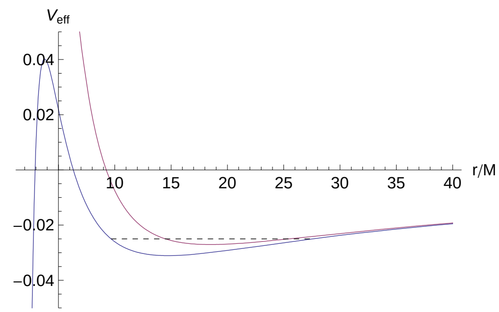
Orbit from the equation of motion

How do orbits come from the equation of motion?

Recall

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{d\,r}{d\,\tau}\right)^2 + V_{\text{eff}}(r)$$

$$V_{\text{eff}}(r) = -\frac{M}{r} + \frac{l^2}{2\,r^2} - \frac{M\,l^2}{r^3}$$



Caption: Effective potential for Einstein's theory (blue) and for Newton's theory (purple). $l = 4.3$ and $e = -0.025$

- **Plot**

How to find the orbit

Use a new variable $u = 1/r$, and substitute $\frac{d\,r}{d\,\tau} = \frac{d\,u}{d\,\phi}\,\frac{d\,\phi}{d\,\tau}\,\frac{d\,r}{d\,u} = \frac{d\,u}{d\,\phi}\,\frac{l}{r^2}\left(-r^2\right)$ to get

$$\boxed{\left(\frac{du}{d\phi}\right)^2 = \frac{e^2-1}{l^2} + \frac{2\,M\,u}{l^2} - u^2 + 2\,M\,u^3.}$$

Integrate to find $u(\phi)$. This needs to be done numerically.

To find $\phi(\tau)$, numerically integrate

$$d\,\tau = \frac{1}{l}\,r(\phi)^2\,d\,\phi$$

Approximation for Mercury (for M/r≪1)

Differentiate to get

$$\frac{d^2u}{d\phi^2} = -M/l^2 - u - 3\,M\,u^2.$$

Which terms are small?

$M\,u$ is of order 1.5km/0.39AU

In[1348]:= `Convert[1.5 Kilo Meter / (0.39 AstronomicalUnit), 1]`

Out[1348]= 2.571×10^{-8}

Q: What is the solution to $\frac{d^2u}{d\phi^2} = -u$?

For $M\,u \ll 1$, we know the solution

$$u(\phi) = \frac{M}{l^2}\left[1 - \epsilon \cos(\phi - \phi_0)\right]$$

Plan: use the Newtonian solution as an approximation, and find the equation with the Newtonian solution substituted in the new term.

$$\frac{d^2u}{d\phi^2} = -M/l^2 - u - 3\,M^3/l^4\left[1 - 2\,\epsilon \cos(\phi - \phi_0) + \epsilon^2 \cos^2(\phi - \phi_0)\right].$$

Q: What does the term $-2\,M^3/l^4$ do?

The term $\cos^2 = [1 + \cos 2\,(\phi - \phi_0)]/2$ drives u at twice the frequency. It does not build up over time.

The term $-6\,\epsilon\,M^3/l^4 \cos(\phi - \phi_0)$ is at the fundamental frequency. It builds over time.

The solution is

$$u(\phi) = \frac{M}{l^2}\left[1 - \epsilon \cos(\phi - \phi_0) - \frac{3\,\epsilon\,M^2}{l^2}\,\phi \sin(\phi - \phi_0)\right]$$

$$= \frac{M}{l^2}\left[1 - \epsilon \cos\left(\phi - \left(\phi_0 + 3\,\frac{M^2}{l^2}\,\phi\right)\right)\right]$$

Verify: $d\,\phi \sin(\phi) = \phi \cos\phi + \sin\phi$.

$d\,(\phi \cos\phi + \sin\phi) = -\phi \sin\phi + 2 \cos\phi$

The term $\phi \sin\phi$ cancels with that term in u.

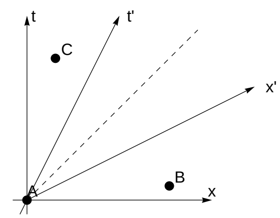
Rewrite

$$u(\phi) = \frac{M}{l^2}\left\{1 - \epsilon \cos\left[\phi - \left(\phi_0 + 3\,\frac{M^2}{l^2}\,\phi\right)\right]\right\}$$

The term $3\,\frac{M^2}{l^2}\,\phi$ means $u(\phi)$ does not repeat every $2\,\pi$. This is the perihelion shift.

Temporal order

Temporal order depends on the frame of reference, if the separation between the two events is spacelike.

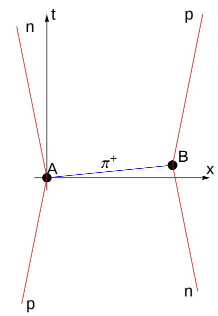


The separation between events A and B is spacelike. Events A and B. Event A occurs before event B in the rest frame (x, t), but in the frame (x', t'), event B occurs before event A.

Q: The separation between events A and C is timelike. Show that event A occurs before event C in every frame.

■ Antiparticles

A proton collides with a neutron. The proton emits a π^+ and becomes a neutron. The neutron absorbs the π^+ and becomes a proton.



The separation between events A and B is spacelike. How can the π^+ travel a spacelike path? The can occur according to Heisenberg.

$$\Delta\,t\,\Delta\,E > \hbar$$

The energy must be of order the mass of the proton.

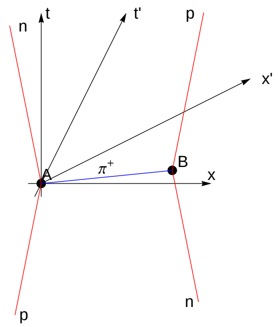
The time is uncertain to

$$\Delta\,t > \frac{\hbar}{m\,c^2}$$

The probability for this to occur is high if

$$\left|(\Delta\,t)^2 - (\Delta\,x)^2\right| < \frac{\hbar^2}{m^2}.$$

As viewed in the rest frame, event A occurs before event B. As viewed in the primed frame, B occurs before A. The neutron emits a π^+ and becomes a proton. Charge is not conserved.



A solution: In the primed frame, the particle is a π^-, not a π^+. For every charged particle, there must be an antiparticle with the opposite charge and the same mass.

- **Plots**