Initialization

Friedman's equation—2 Feb

- Announcements
  - Homework 3 is due on 2/6. The link is on the syllabus on angel. I fixed a bug in #1.
  - Student Book Store will soon return unsold copies of Hartle.
- Outline
  - Redshift derived from the symmetry
  - Friedman's equation
  - First Law of thermodynamics

Redshift derived from isotropy

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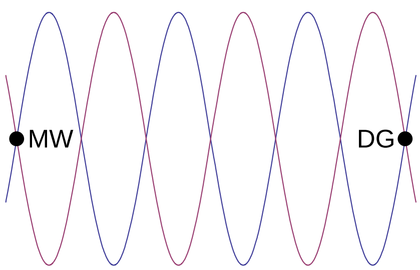


Space is permeated with standing waves of light. (To get a standing wave, waves goes in opposite directions.)

At some time 1, the Milky Way and a distant galaxy are N waves apart. (Here $N = 3$.) Later, MW and DG have moved apart by the factor a_2/a_1. Does the MW move to the right or the left of the node that it was on? Since there are no special directions, the MW has to stay on the node. Same for DG. Therefore

$$D_1 = N\,\lambda_1$$
$$D_2 = N\,\lambda_2$$
$$\frac{a_2}{a_1} = \frac{D_2}{D_1} = \frac{\lambda_2}{\lambda_1}$$

The wavelength of light expands by the same factor as does the universe.

More normal form: a_r at reception is 1. a_e at emission is written as a.

$$\lambda_r = \lambda_e/a.$$

Redshift is defined to be $z = (\lambda_r - \lambda_e)/\lambda_e$.

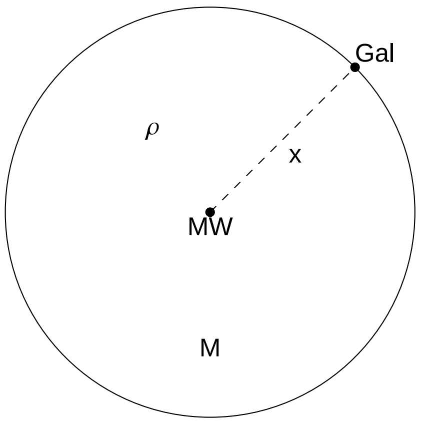
Then

$$z = a^{-1} - 1$$
$$a = (1+z)^{-1}$$

- **Plot**

Friedman's equation

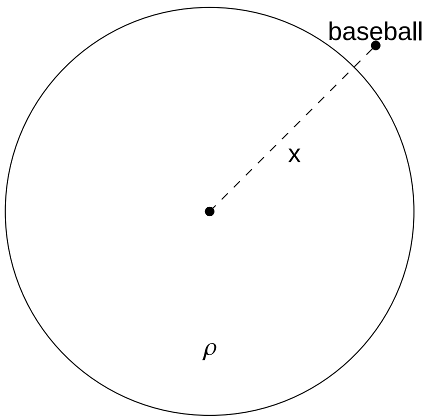
Friedman's equation relates the expansion of the universe and mass density, expansion rate, and curvature.



Consider a big sphere centered on the Milky Way. There is a galaxy on the surface of the sphere. The sphere grows as the universe expands. Its distance is x, and its comoving coordinate is r. There is a total mass M inside the sphere. The mass density is ρ.

We want to know the position of the galaxy vs time. Since the galaxy is a proxy for the universe, we will find out how much the universe expands with time.

We already know how to solve this problem. Change the names.



Q: How do you find the acceleration of the baseball?
What determines whether the baseball escapes from Earth?

The mass inside is slow stuff: $v \ll 1$. Galaxies are slow, since $v \sim 300\ \mathrm{km/s} = 0.001$.

The energy of the galaxy (of unit mass) is

$$\frac{1}{2} v^2 - \frac{G M}{x} = \text{constant.}$$

Insert Hubble's Law $v = H\, x$ and $H = \frac{1}{a} \frac{d a}{d t}$

Insert $M = \frac{4 \pi}{3} \rho\, x^3$ to get

$$\left(\frac{1}{a} \frac{d a}{d t}\right)^2 - \frac{8 \pi}{3} G \rho = \frac{\text{constant}}{r^2 a^2}$$

Since the LHS does not depend on which galaxy I pick, the RHS should be independent of r. The constant must depend on r^2.

$$\boxed{\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3} G\rho = \frac{\text{constant}}{a^2}}$$

- **Plot**

What is the constant? It comes from E's field equation

Here we have to use a result that we do not have until we have Einstein's field equation. The constant is $-1/r_0^2$. It is related to the radius of curvature in the metric. This is Friedman's equation:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3} G\rho = -\frac{1}{r_0^2 a^2}$$
$$H^2 - \frac{8\pi}{3} G\rho = -\frac{1}{r_0^2 a^2}$$

Interpretation?

KE + PE = Total

The KE term became H^2.

The PE term became $G\rho$.

Q: Why is KE related to H^2 and PE related to $G\rho$?

The total energy term became $-\frac{1}{r_0^2 a^2}$.

Q: State this in a way that points out the surprise in this result.

Curvature is related to mass density

Define H_0 to be Hubble's constant at the present time, and ρ_0 to be the mass density at the present time.
At the present time,

$$H_0^2 - \frac{8\pi}{3} G\rho_0 = -\frac{1}{r_0^2}$$
$$(r_0 H_0)^2 = \left(\frac{8\pi G\rho_0}{3H_0^2} - 1\right)^{-1}$$

Interpretation: H_0^{-1} is a length, called the Hubble length. The radius of curvature r_0 compared to the Hubble length is the RHS. The RHS involves the mass density and Hubble's constant.

The quantity is called the density parameter. $\Omega_0 \equiv \frac{8\pi G\rho_0}{3H_0^2}$.

Interpretation: Ω is the ratio of the $|\text{PE}|$ to the KE.

$$(r_0 H_0)^{-2} = \Omega_0 - 1$$

Suppose r_0 is infinity. Then the density parameter equals the critical density $\rho_0 = \frac{3H_0^2}{8\pi G}$.

Suppose $r_0^2 > 0$. Then $\Omega_0 > 1$, and the density is greater than the critical density.
Suppose $r_0^2 < 0$. Then $\Omega_0 < 1$, and the density is less than the critical density.

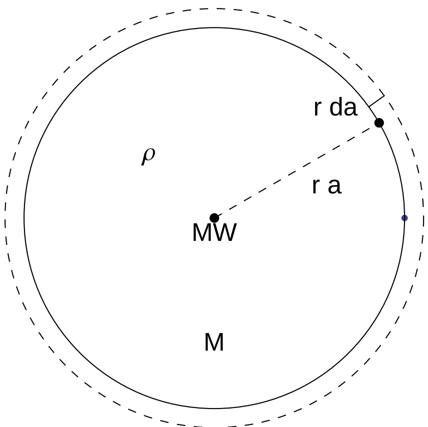
Finding $a(t)$

We want to find $a(t)$, which means to integrate Friedman's equation,

$$\left(\frac{1}{a}\ \frac{da}{dt}\right)^2 - \frac{8\pi}{3}\, G\,\rho = -\frac{1}{r_0^2\, a^2}$$

How does density change with expansion parameter?
Q: I have a box of marbles. If the box expands with the universe by a factor of 2, by what factor does the density of marbles change?

"Matter" with and without pressure



The energy inside the sphere changes as the universe expands because the inside does work on the outside. Energy density is ρ.

$$d\,E = -p\,A\,d\,x = -p\,d\,V$$
$$d\,\rho\,a^3 = -p\,d\,a^3.$$

- **Pressureless matter**

The speed of galaxies is 300km/s=0.001. The pressure is 10^{-6}.
Q: The pressure of galaxies is small compared with what?

$$d\left(\rho\,a^3\right) = 0$$
$$\boxed{\rho \sim a^{-3}}$$

- **Radiation has pressure**

For a photon, $E^2 = p^2 + m^2$ becomes $E = |\,p\,|$. The pressure is proportional to the momentum p_x times the rate at which photons are hitting p_x/E. Here, p is momentum. The pressure in one direction is $p_x^2\big/E = \frac{1}{3}\,E$.

Therefore (p is pressure now)

$$p = \frac{1}{3}\,\rho$$

Put that in

$$d\,\rho\,a^3 = -p\,d\,a^3$$

to get

$$\boxed{\rho \sim a^{-4}}$$

Q: Simplicio says, “Radiation can be considered to be photons. I can think of photons as marbles. The energy density should then scale as a^{-3}, just as it does for marbles.” In what way is Simplicio wrong?

Vacuum

The vacuum has energy density from the quantum mechanics. The simpliest model is that a field is a harminic oscillator. The ground state of which is $\frac{1}{2} \hbar \omega$. Each field has some "vacuum energy." This energy density is a property of the field and therefore does not depend on the expansion of the universe.

$$\rho \sim \text{constant}$$

Then the pressure is negative.

$$d \rho\, a^3 = \rho\, d a^3.$$
$$\rho\, d a^3 = -p\, d a^3$$
$$p = -\rho$$

The pressure of the material inside the sphere is pulling on the outside, not pushing.

For pressureless matter
$$\rho \sim a^{-3}.$$
For radiation
$$\rho \sim a^{-4}.$$
For the vacuum
$$\rho \sim \text{constant}.$$

Fig