13 Mar 2012—Equivalence Principle. Einstein's path to his field equation

15 Mar 2012—Tests of the equivalence principle

20 Mar 2012—General covariance. Math. Covariant derivative

- Homework 6 is due on 27th.
- Outline
  - General covariance
  - Mathematics: scalars, vectors, tensors.
  - Covariant derivative

Lorentz transformation

Lessons from special relativity:
1) Laws of physics must be written a scalers, vectors, or tensors. They cannot be parts of a vector.
2) Scalars, vectors, and tensors are defined by their transformation properties.

Consider the space-time coordinates x^μ. In a different coordinate system, the coordinates are x'^μ. The Lorentz transformation is

$$x'^\alpha = \Lambda^\alpha{}_\beta\, x^\beta + a^\alpha.$$

The Lorentz transformation satisfies

$$\Lambda^\alpha{}_\gamma\, \Lambda^\beta{}_\delta\, \eta_{\alpha\beta} = \eta_{\gamma\delta},$$

where

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What is special about a Lorentz transformation? The length of $d x^\alpha$ is defined to be

$$-d t^2 + \overrightarrow{d x}^2 = \eta_{\alpha\beta}\, d x^\alpha\, d x^\beta.$$

In a different frame,

$$-d t'^2 + \overrightarrow{d x'}^2 = \eta_{\alpha\gamma}\, d x'^\alpha\, d x'^\gamma.$$
$$= \eta_{\alpha\gamma}\, \Lambda^\alpha{}_\beta\, \mathrm{dx}^\beta\, \Lambda^\gamma{}_\delta\, d x^\delta$$
$$= \eta_{\beta\delta}\, d x^\beta\, d x^\delta$$
$$= -d t^2 + \overrightarrow{d x}$$

Because the laws of physics are not invarient under all Lorentz transformations, let us impose

$$\Lambda^0{}_0 \geq 1$$

and

$$\det \Lambda = +1.$$

Q: What transformations do these restrictions forbid?

These transformations are called proper, inhomogeneous Lorentz transformations.
Proper means with the restrictions.
Inhomogeneous means $a^\alpha \neq 0$.

Equivalence Principle—Principle of General Covariance

Equivalence Principle

Because for an observer falling freely from the roof of a house there exists—at least in his immediate surroundings— no gravitational field—Einstein.

In the vicinity of any point in a gravitational field, it is possible to find an inertial frame such that the laws of physics are the same as those without gravity.

Example: To treat gravity, use an inertial frame. Write the law of physics in this frame. Then transform back to the original frame.
The equation of motion

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma = 0$$

preserves its form under a general coordinate transformation. In the absence of gravity, it is

$$\frac{du^\alpha}{d\tau} = 0,$$

which is true.

Restated: To eliminate gravity, transform to a gravity-free frame. To account for gravity, transform back to the original frame.

Principle of General Covariance

Principle of General Covariance: A statement of physics is true in a gravitational field if
1) it is true in the absence of gravity, and
2) it preserves its form under a general coordinate transformation.

Equivalence Principle ⇒ Principle of General Covariance
Suppose an equation where gravity is present and conditions (1) and (2) are true. Condition (2) says the equation is true in any frame if it is true in one frame. Condition (1) says the equation is true in the absence of gravity. Therefore the equation satisfies the EP and is therefore true.

What is the purpose of a general coordinate transformation? It is a way to define the physics of gravity. It is an example of a symmetry that define gravity.

What is the purpose of Lorentz invariance? It limits the allowed equations of all physics.
Example: Transform Newton’s 2nd Law with the Lorentz transformation. The result has the velocity of the new coordinate system with respect to the old one. The Principle of Relativity, which says the velocity cannot appear, rejects Newton’s 2nd Law.

We will write equations that satisfy (2) in this way: Use scalars, vectors, and tensors, and because we know the transformation laws, (2) is satisfied.

Scalars, vectors, and tensors

Scalar

Scalars do not change under a general coordinate transformation.

Vector

A contravariant vector transforms in the same way as dx^α.

$$dx'^\mu = dx^\nu \frac{\partial x'^\mu}{\partial x^\nu}$$

$$A'^{\mu} = A^{\nu} \frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

A covariant vector transforms as

$$A'_{\mu} = A_{\nu} \frac{\partial x^{\nu}}{\partial x'^{\mu}}$$

Q: Is the derivative of a scalar $\frac{\partial \phi}{\partial x^{\mu}}$ a contravariant or covariant vector?

- **Tensor**

A tensor $T_{\mu}{}^{\nu}$ transforms like $A_{\mu} B^{\nu}$. A tensor $T_{\mu\nu}$ transforms like $A_{\mu} B_{\nu}$, etc.

Prove that the metric tensor is a tensor.
1) There is a frame in which the metric tensor is $\eta_{\alpha\beta}$. The position vector is ξ^{α}.
Q: Reason?
2) For a general coordinate frame x^{α},

$$g_{\mu\nu} \equiv \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}}.$$

3) In a different coordinate frame x'^{α},

$$g'_{\mu\nu} \equiv \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x'^{\mu}} \frac{\partial \xi^{\beta}}{\partial x'^{\nu}}$$

$$= \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}}$$

Q: Reason?

4) $$g'_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}}$$

$$= g_{\sigma\rho} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\nu}}$$

Q: Reason?
Q: Is $g_{\mu\nu}$ a covariant or contravariant or mixed tensor?

Tensor algebra

- **Linear combination**

If a and b are scalars, and $A^{\mu\nu}$ and $B^{\mu\nu}$ are tensors, then

$$a A^{\mu\nu} + b B^{\mu\nu}$$

is a tensor.
Q: How do you prove this?

- **Direct product**

If $A^{\alpha\beta}$ and $B^{\mu\nu}$ are tensors, then

$$A^{\alpha\beta} B^{\mu\nu}$$

is a tensor.

- **Contraction**

If $A^{\alpha\beta}$ and $B_{\mu}{}^{\nu}$ are tensors, then

$$T^{\alpha\nu} \equiv A^{\alpha\beta} B_{\beta}{}^{\nu}$$

is a tensor.

Special but important case: The contraction of a time-like vector dx^α with itself. The proper time $d\tau^2$ of a time-like vector is

$$d\tau^2 = -dx_\alpha\, dx^\alpha.$$

Mathematician’s lingo

We used contravarviant vectors A^μ and covariant vectors A_μ. Mathematicians call them something else and they have different mental pictures.

Contravarviant vectors A^μ are called vectors. Picture them as arrows.

Covariant vectors A_μ are called “dual vectors” or “one-forms”.

We call $A^\mu B_\mu$ the contraction of the contravariant vector A^μ and the covarient vector B_μ.

Mathematicians say a dual vector is a linear function of vectors to real numbers.
Q: Using our language, explain how this is so.

Mathematicians picture a dual vector as parallel sheets of paper. $A^\mu B_\mu$ is the number of sheets of paper pierced by A^μ.
Q: If you want to make A^μ is bigger, what do you do to the arrows? If B_μ is bigger, what do you do to the sheets of paper?

All operations must occur at the same point. For example, adding two tensor fields is done at each point.

$$a\, A(x)^{\mu\nu} + b\, B(x)^{\mu\nu}$$

is legitimate, but

$$a\, A(x)^{\mu\nu} + b\, B(x')^{\mu\nu}$$

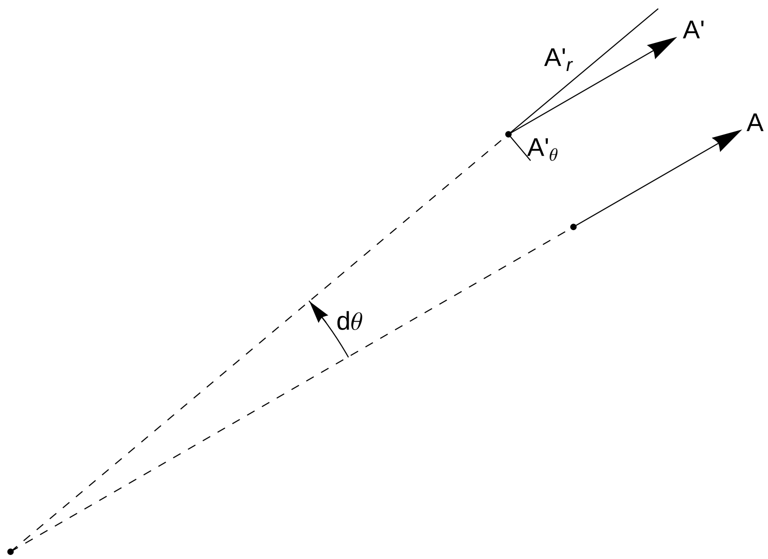
is not necessarily a tensor when x and x' are distinct points.

Mathematicians speak of a "tangent space" at each point x. This the the space in which vectors, 1-forms, and tensors exist. The tangent space at different points may be distinct.

Example: The surface of Earth. The tangent space at the equator has vectors, 1-forms, and tensors. The vectors are arrows parallel to the surface of Earth. The vectors in the tangent space at the north pole are parallel to the surface of Earth at the north pole. The vectors in the two tangent spaces are different.

Covariant derivative of a contravariant vector

How do you take derivatives of tensors?
Requirements
1) The derivative of a tensor must be a tensor
2) The derivative must measure a physical quantity and not merely a quirk of the coordinate system.



Example of a quirk of the coordinate system: A constant vector field A^μ in two dimensions with polar coordinates. Even though A is constant, $\frac{\partial A^r}{\partial \theta}$ is not zero.

Mathematical problem: The derivative means to compare $A(x + dx)$ and $A(x)$.

We already found that the equation of motion is

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma = 0.$$

The terms $\frac{du^\alpha}{d\tau}$ and $\Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma$ are not tensors. Proof: $\Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma$ is zero in a gravity-free frame. If it were a tensor, it must be zero in all frames.

We derived the equation of motion by differentiating the 4-velocity.
Rewrite

$$\frac{du^\alpha}{d\tau} = \frac{dx^\beta}{d\tau}\frac{\partial u^\alpha}{\partial x^\beta} = u^\beta \frac{\partial u^\alpha}{\partial x^\beta}$$

and insert to get

$$u^\beta\left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, u^\gamma\right) = 0.$$

This says: In the parenthesis is the change in u^α in the β direction. Contracting it (taking the dot product) with u^β results in 0.

Contraction is a tensor operation. Therefore $\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, u^\gamma$ is a tensor.

For any contravarient vector A^α,

$$\boxed{\nabla_\beta A^\alpha = \frac{\partial A^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, A^\gamma}$$

is a tensor. This is called the covariant derivative. Another notation:

$$\boxed{A^\alpha{}_{;\beta} = A^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma}\, A^\gamma}$$

Q: Is $A^\alpha{}_{;\beta} \equiv \nabla_\beta A^\alpha$ covariant or contravarient in the index β?

Example: For 2-dimensional polar coordinates, the metric is

$$ds^2 = dr^2 + r^2\, d\theta^2$$

The non-zero Christoffel symbols are (8.17)

$\Gamma^r_{\theta\theta} = -r$

$\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = 1/r.$

$A^r{}_{;r} = A^r{}_{,r}$

$A^r{}_{;\theta} = A^r{}_{,\theta} - r\, A^\theta$

$A^\theta{}_{;r} = A^\theta{}_{,r} + 1/r\, A^\theta$

$A^\theta{}_{;\theta} = A^\theta{}_{,\theta} + 1/r\, A^r$

The covariant derivative of the r component in the r direction is the regular derivative. If a vector field is constant, then $A^r{}_{;r} = 0$. The covariant derivative of the r component in the θ direction is the regular derivative plus another term. Even if a vector field is constant, $A^r{}_{;\theta} \neq 0$. The Γ term accounts for the change in the coordinates.

The idea of a covariant derivative of a vector field A in the direction a. Is this a good definition?

$\nabla_a A^\alpha = \lim_{\epsilon\to 0} \frac{1}{\epsilon} [A(x + \epsilon\, a) - A(x)]$???

However, the components of $A(x + \epsilon\, a)$ may be different even if the vector is the same, because the coordinates are changing. We must move $A(x + \epsilon\, a)$ back to x before comparing. Moving is called parallel transporting. This is what the Γ term does.

$\nabla_a A^\alpha = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \{\text{parallel transport}[A(x + \epsilon\, a)] - A(x)\}$

Q: Simplicio: Covariant derivatives are irrelavant. I want to know about gravity. In what way is Simplicio mistaken?

- **Plot**

22 Mar 2012—Riemann-Christoffel curvature tensor. Bianchi Identity

27 Mar 2012—Stress-energy tensor. Conservation of energy and momentum. Einstein’s discovery of the field equation