13 Mar 2012—Equivalence Principle. Einstein’s path to his field equation

15 Mar 2012—Tests of the equivalence principle

20 Mar 2012—General covariance. Math. Covariant derivative

22 Mar 2012—Riemann-Christoffel curvature tensor.

27 Mar 2012—Bianchi’s identity. Stress-energy tensor. Conservation of energy and momentum.

- Ricci tensor and curvature scalar
- Bianchi’s identity
- Stress energy tensor $T^{\mu\nu}$
  - Stress energy tensor of a particles
  - Stress energy tensor of a perfect gas
- Energy and momentum conservation $\nabla_\nu T^{\mu\nu} = 0$
- Bianchi’s identity is related to energy and momentum conservation

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Ricci tensor and curvature scalar, symmetry

The Ricci tensor is a contraction of the Riemann-Christoffel tensor

$$R_{\gamma\beta} \equiv R^{\alpha}{}_{\gamma\alpha\beta}.$$

The curvature scalar is the contraction of the Ricci tensor

$$R = g^{\beta\gamma} R_{\gamma\beta}.$$

Symmetry properties of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta} \equiv g_{\alpha\sigma} R^{\sigma}{}_{\beta\gamma\delta}$

1) Symmetry in swapping the first and second pairs

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

2) Antisymmetry in swapping first pair or second pair

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$$

3) Cyclicity in the last three indices.

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$$

Example: Curvature scalar for surface of a 2-d sphere

The metric is

$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

The nonzero parts of the Christoffel symbol are

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = -\sin\theta\cos\theta$$

The Riemann-Christoffel tensor is in general

$$R^{\sigma}{}_{\gamma\alpha\beta} = \frac{\partial}{\partial x^{\alpha}}\Gamma^{\sigma}{}_{\gamma\beta} - \frac{\partial}{\partial x^{\beta}}\Gamma^{\sigma}{}_{\gamma\alpha} + \Gamma^{\sigma}{}_{\alpha\epsilon}\Gamma^{\epsilon}{}_{\gamma\beta} - \Gamma^{\sigma}{}_{\beta\epsilon}\Gamma^{\epsilon}{}_{\gamma\alpha}$$

Q/NA: Compute one non-zero component (no sum)

$R^{\theta}{}_{\phi\theta\phi} = \ldots = \sin^2\theta$

Q/NA: Compute (no sum)

$R^{\theta\phi}{}_{\theta\phi}$

Q/NA: Compute the Ricci tensor. Answer:

$R^{\theta}{}_{\theta} = R^{\phi}{}_{\phi} = a^{-2}$

$R^{\theta}{}_{\phi} = R^{\phi}{}_{\theta} = 0$

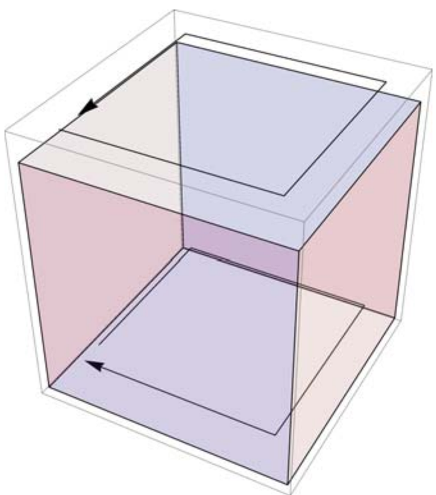
Compute the curvature scalar R. Answer $R = 2\,a^{-2}$

Q: What information is in the curvature scalar?

Bianchi identity

Bianchi's identity: The curvature induced change of a vector carried over the 6 faces of a cube is zero. (Carry in an oriented way, so that the right-handed direction points out.)

Proof: Each side is traversed twice in opposite directions. Therefore the total change is zero.



Q: The x-y planes have been traversed in the figure. Draw the traversal of the y-z planes

In equation form:

The change on the y-z at face at x is

$$dA_\sigma(x) = -A_\sigma R^{\sigma}{}_{\gamma y z}(x)\, dy\, dz$$

The change on the y-z face at $x + dx$ is

$$dA_\sigma(x+dx) = -A_\sigma R^{\sigma}{}_{\gamma y z}(x+dx)\, dy\, dz$$

The change over both faces is

$$dA_\sigma(x+dx) - dA_\sigma(x) = -A_\sigma \nabla_x R^{\sigma}{}_{\gamma y z}\, dx\, dy\, dz$$

Q: Is $\nabla_x R^{\sigma}{}_{\gamma y z}$ the same as $\frac{\partial}{\partial x} R^{\sigma}{}_{\gamma y z}$?

Traverse the face at $x + dx$ in the outward-pointing sense and the face at x in the outward-pointing sense.

The change over all 6 faces is

$$A_\sigma\, dx\, dy\, dz \left(\nabla_x R^{\sigma}{}_{\gamma y z} + \nabla_y R^{\sigma}{}_{\gamma z x} + \nabla_z R^{\sigma}{}_{\gamma x y}\right)$$

and since each side in traversed in opposite directions, it is zero.

We chose x, y, and z, but we could have also chosen t for one of the directions. Therefore, we have proved the Bianchi identity,

$$\boxed{\nabla_\alpha R^\sigma{}_{\tau\beta\gamma} + \nabla_\beta R^\sigma{}_{\tau\gamma\alpha} + \nabla_\gamma R^\sigma{}_{\tau\alpha\beta} = 0}$$

A contracted form of the Bianchi identity is:

$$\boxed{\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = 0}$$

- **Fig**

Stress-energy tensor without gravity

Definition of the stress-energy tensor $T^{\alpha\beta}$.
1) Let u^β be the 4-velocity of the observer. Then

$$T^\alpha{}_\beta u^\beta = T_\beta{}^\alpha u^\beta = -d p^\alpha / d \text{volume}$$

is the density of 4-momentum. $-T^\alpha{}_\beta u^\beta \, dx \, dy \, dz$ is the 4-momentum in a box.
Q: Let n^α be a unit vector. What is $T_{\alpha\beta} u^\beta n^\alpha$?
2) Let i and j be indices in space. $T_{ij} = T_{ji}$ is the force in the i direction on a unit surface perpendicular to the j direction. It is also the force in the j direction on a unit surface perpendicular to the i direction.
Q: What is T_{xx}?

- **Stress energy tensor for a swarm of particles**

The particles have mass m and 4-velocity u. Their momentum is $p = m u$. There are n particles per unit volume in the frame of the particles.

In a frame in which the particles are moving, the flux of particles is

$$s = n u.$$

The x component of s is the number of particles per second crossing a unit area perpendicular to the x-direction.

$s^0 = n \left(1 - v^2\right)^{-1/2}$.
Q: What is the reason for the factor $\left(1 - v^2\right)^{-1/2}$?

Since each particle carries momentum p, the density of 4-momentum is

$$T^{\alpha 0} = p^\alpha s^0$$
$$= m u^\alpha n u^0$$

and the flux of 4-momentum is

$$T^{\alpha i} = p^\alpha s^i$$
$$= m u^\alpha n u^i$$

All together,

$$T^{\alpha\beta} = m n u^\alpha u^\beta.$$

- **Stress energy tensor for a perfect gas**

Consider the frame in which the gas is at rest.

The T^{00} term is the sum of $m n u^0 u^0$. $m u^0$ is the mass-energy of the particle. $n u^0$ is the number density. The product is the mass-energy density ρ.

The T^{xx} term is the sum of $m\, n\, u^x\, u^x$. What is this?

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

where ρ is the mass-energy density and P is the pressure.

Q: There are gas particles moving in the x and y directions. Why don't they transfer momentum across the y-z plane in the y direction?

In some other frame, let u be the 4-velocity of the gas. In this frame,

$$T^{\alpha\beta} = (\rho + P)\, u^\alpha\, u^\beta + P\, \eta^{\alpha\beta}.$$

ρ and P are the mass-energy density and pressure in the rest frame; they are scalars. This is clearly a tensor.
Check that it is correct in the frame in which the fluid is at rest: $u^\alpha = (1, 0, 0, 0)$.

$$T^{00} = (\rho + P)\,(1)\,(1) + P(-1) = \rho.$$

$$T^{11} = (\rho + P)\,(0)\,(0) + P(1) = P.$$

Conservation of energy and momentum

The divergence of the stress-energy tensor is 0

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = \frac{\partial}{\partial t} T^{\alpha 0} + \frac{\partial}{\partial x} T^{\alpha x} + \frac{\partial}{\partial y} T^{\alpha y} + \frac{\partial}{\partial z} T^{\alpha z} = 0$$

Interpretation: Integrate this inside a fixed 3-d box

$$\int \left(\frac{\partial}{\partial t} T^{\alpha 0} + \frac{\partial}{\partial x} T^{\alpha x} + \frac{\partial}{\partial y} T^{\alpha y} + \frac{\partial}{\partial z} T^{\alpha z} \right) dx\, dy\, dz$$

$$= \frac{\partial}{\partial t} \int T^{\alpha 0}\, dx\, dy\, dz + \int T^{\alpha x}(x + dx)\, dy\, dz - \int T^{\alpha x}(x)\, dy\, dz + \ldots = 0$$

We said that $T^{\alpha 0}$is the density of the α component of the 4 momentum, and $T^{\alpha x}$ is the flux of the the α component of the 4 momentum in the x direction $\left(d\, p^\alpha / s / m^2\right)$

The change in the amount of p^α inside the box + the amount going out through the surfaces is zero.

Stress-energy tensor with gravity

Q: What is the rule for including gravity?

Energy-momentum conservation is

$$\nabla_\alpha T^{\alpha\beta} = 0.$$

For a perfect gas,

$$T^{\alpha\beta} = (P + \rho)\, u^\alpha\, u^\beta + P\, g^{\alpha\beta}.$$

Conservation of energy and momentum is related to Bianchi’s identity

Energy-momentum conservation is

$$\nabla_\alpha T^{\alpha\beta} = 0.$$

A contracted form of the Bianchi identity is:

$$\boxed{\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu}\, R\right) = 0}$$

Q: Is energy-momentum conservation is a principle of physics or geometry? Is the Bianchi identity is a principle of physics or geometry?

Einstein’s Field Equation is

$$R^{\mu\nu} - \frac{1}{2}\, R\, g^{\mu\nu} = -8\, \pi\, G\, T^{\mu\nu}$$

Q: In light of Einstein’s equation, what is surprising about the conservation of energy and momentum?

29 March—Einstein’s discovery of the field equation