13 Mar 2012—Equivalence Principle. Einstein’s path to his field equation

15 Mar 2012—Tests of the equivalence principle

20 Mar 2012—General covariance. Math. Covariant derivative

22 Mar 2012—Riemann-Christoffel curvature tensor.

27 Mar 2012—Bianchi’s identity. Stress-energy tensor. Conservation of energy and momentum.

29 March—Einstein’s discovery of the field equation

- Derivation of the Field Equations
- Cosmological constant
- Einstein’s toy

"Derivation" of Einstein's field equation

E’s plan was to write

measure of curvature = source of gravity.

- **Einstein’s happiest thought, Nov 1907**

- **Mathematics of curvature 1913**

Einstein & Grossman, Z. Math. Physik, 62, 225, (1913) the mathematics of curvature.

- **The source of gravity is the stress-energy tensor**

A guess for the source of gravity is the stress-energy tensor. In the limit of slow speeds, the stress-energy tensor is

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \longrightarrow \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Q: Why does $P \to 0$ for slow speeds?
The mass density is the source of gravity for Newton's gravity.

- **Choices for the measure of curvature**
  11 Nov 1915, Prussian Acad. Wissen., p 799

We have many choices for a measure of curvature. For the LHS to equal the RHS, we have to use the same rank for both.
Q: What are the choices?
Q: What rank 2 tensor is a measure of curvature?

For slowly moving particles, we reasoned using the equivalence principle.
The equation of motion is

$$\frac{d\,u^i}{d\tau} + \Gamma^i_{\nu\alpha}\, u^\nu\, u^\alpha = 0$$

The biggest term is u^0. The equation of motion is

$$\frac{d u^i}{d\tau} + \Gamma^i_{00} = 0.$$

We computed Γ to find

$$\frac{du^i}{dt} + \frac{1}{2}\frac{\partial g_{00}}{\partial x^i} = 0.$$

Newtons' equation is

$$\frac{du^i}{dt} + \frac{\partial \phi}{\partial x^i} = 0$$

Therefore

$$g_{00} = -(1 + 2\,\phi).$$

We know $\nabla^2 \phi = 4\,\pi\,\mathrm{G}\rho$.
Therefore the constant is $-8\,\pi\,G$.

Here was Einstein's guess:

$$R^{\mu\nu} = -8\,\pi\,G\,T^{\mu\nu}.$$

- **Big success Nov 18, 1915, PAW, p. 831.**

Calculate precession of perihelion of Mercury.
Calculate bending of light.

- **Big problem**

Energy and momentum conservation means $\nabla_\nu T^{\mu\nu} = 0$. However $\nabla_\nu R^{\mu\nu} \neq 0$.

Einstein tried several patches:
(1) Give up on general invariance.

We know about Bianchi's identity (that carrying a vector around the 6 faces of a cube yields 0), but Einstein didn't.

- **Success. Field equation. Nov 25, 1915. PAW, p 844.**

$$R^{\mu\nu} - \frac{1}{2}\,R\,g^{\mu\nu} = -8\,\pi\,G\,T^{\mu\nu}$$

- **Why was E able to calculate the bending of light and the precession of Mercury with the wrong equation?**

You can show

$$R_{\mu\nu} = -8\,\pi\,G\left(T_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,T\right)$$

where T is the contraction T^μ_μ.

Outside of the sun, there is no mass. $T_{\mu\nu} = 0$. Therefore $R_{\mu\nu} = 0$ and certainly $R = 0$. The missing term is zero where light travels and where Mercury travels.

The cosmological constant

Friedmann showed that the universe is not static.

Q: How did Friedmann show this?

Einstein thought that the universe must be unchanging and everlasting. He invented the cosmological constant.

Einstein's argument:
The equivalence principle says there exists a frame in which the effects of gravity vanish. In that frame the metric is $\eta^{\alpha\beta}$ and first derivatives of $\eta^{\alpha\beta}$vanish. (The first derivatives give nonzero Christoffel symbols.) $\frac{\partial}{\partial x^\alpha}\eta^{\alpha\beta} = 0$. The transformation of the zero tensor is zero. Therefore

$$\nabla_\alpha g^{\alpha\beta} = 0.$$

A stress-energy tensor having the form

$$T^{\alpha\beta} = -\Lambda\, g^{\alpha\beta},$$

where Λ is a constant, is conserved. E: I can think of no reason why such a such a stress-energy tensor cannot exist.
Pauli: What is not forbidden is mandatory.

What pressure and mass-energy density does this imply? In the frame in which the gas is at rest,

$$T^{\alpha\beta} = -\Lambda\, g^{\alpha\beta} = (P + \rho)\, u^\alpha\, u^\beta + P\, g^{\alpha\beta}$$

becomes

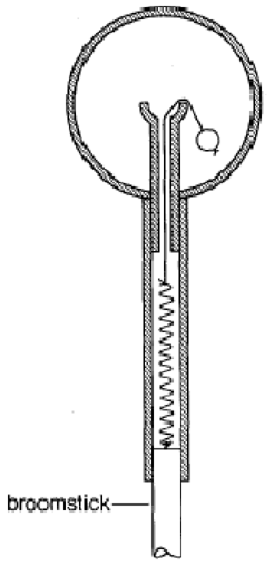
$$\begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda & 0 & 0 \\ 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & -\Lambda \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

The pressure $P = -\rho$.

Einstein's toy

"While I was living in Princeton, my wife and I would from time to time take a small puzzle involving physics to our neighbour Professor Einstein often as a birthday present. The last of these, presented on his seventy-sixth birthday, was, I believe, original. It was derived from an old-fashioned toy for small children: a ball on a string is tied to a cup in which the child has to catch the ball. But our modification was for Einstein a problem which he enjoyed, and solved at once.

A metal ball attached to a smooth thread is enclosed in a transparent globe. There is a central, transparent, cup in which the ball could rest; but initially the ball hangs by the thread outside the cup (as shown in the diagram). The thread runs from the ball up to the rim of the cup and down through a central pipe. Below the globe the thread is tied to a long, weak, spiral spring protected by a transparent tube which ends in a long pole broom-handle."
—Eric Rogers, in *Einstein, A Centenary Volume*, A. P. French, ed., Harvard, 1979, p. 131.



Q: How do you get the ball in the cup?
Q: How is this related to $G_{\mu\nu} = -8\pi G\, T_{\mu\nu}$?