3 Apr 2012—Derivation of the Schwartzschild and Robertson-Walker metrics

5 Apr 2012—Friedmann's equation

- Derivation of Friedmann's equation

Outline

Summary of what we have done

The metric is

$$ds^2 = -dt^2 + a(t)^2 \left(\tilde{g}_{rr}\, dr^2 + \tilde{g}_{\theta\theta}\, d\theta^2 + \tilde{g}_{\phi\phi}\, d\phi^2\right)$$

where $\tilde{g} = \begin{pmatrix} \left[1-(r/r_0)^2\right]^{-1} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$

Einstein's Field Equation is

$$\boxed{R_{\mu\nu} = -8\pi G S_{\mu\nu}},$$

where the source is

$$\boxed{S_{tt} = \tfrac{1}{2}(\rho + 3P)}$$

$$\boxed{S_{ij} = \tfrac{1}{2}(\rho - P)\, a^2\, \tilde{g}_{ij}}$$

What's left

Compute the Ricci tensor. Results

$$\boxed{R_{00} = 3\,\frac{\ddot{a}}{a}}$$

$$\boxed{R_{ii} = -\left(\ddot{a}\; a + 2\,\dot{a}^2 + 2\,r_0^{-2}\right)\tilde{g}_{ii}}$$

Use Einstein's Field Equations.

Q: What Newtonian quantity is in R_{00}?

Computation of Ricci tensor

Plan:
a. Compute R_{tt}.
b. Compute $R_{rr}$to find $f(t)$.

a) $R_{00} = -\Gamma^{\alpha}{}_{00,\alpha} + \Gamma^{\alpha}{}_{0\,\alpha,0} + \Gamma^{\alpha}{}_{\sigma 0}\,\Gamma^{\sigma}{}_{0\,\alpha} - \Gamma^{\alpha}{}_{\sigma\alpha}\,\Gamma^{\sigma}{}_{00}$
1st and 4th terms are zero because $\Gamma^{\alpha}{}_{00} = 0$.

2nd term: $\Sigma_i \Gamma^i{}_{0\,i,0} = \Sigma_i \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = 3\left(\frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^2\right)$

3rd term: If $\alpha = 0$, it is 0. $\Sigma_{i,j} \Gamma^i{}_{j0} \Gamma^j{}_{0\,i} = \Sigma_i (\Gamma^i{}_{i0})^2 = 3\frac{\dot{a}}{a}$

Therefore

$$\boxed{R_{00} = 3\frac{\ddot{a}}{a}}$$

b) $R_{rr} = -\Gamma^\alpha{}_{rr,\alpha} + \Gamma^\alpha{}_{r\,\alpha,r} + \Gamma^\alpha{}_{\sigma\,r} \Gamma^\sigma{}_{r\,\alpha} - \Gamma^\alpha{}_{\sigma\alpha} \Gamma^\sigma{}_{rr}$

$= -\Gamma^t{}_{rr,t} + \Gamma^t{}_{r\,t,r} + \Gamma^t{}_{\sigma\,r} \Gamma^\sigma{}_{r\,t} - \Gamma^t{}_{\sigma\,\alpha} \Gamma^\sigma{}_{r\,r}$

$= -(+\Gamma^r{}_{rr,r} + \Gamma^\theta{}_{rr,\theta} + \Gamma^\phi{}_{rr,\phi})$

$+(\Gamma^r{}_{r\,\alpha,r} + \Gamma^\theta{}_{r\,\theta,r} + \Gamma^\phi{}_{r\,\phi,r}) + \Gamma^\alpha{}_{\sigma\,r} \Gamma^\sigma{}_{r\,\alpha} - \Gamma^\alpha{}_{\sigma\alpha} \Gamma^\sigma{}_{rr}$

Collect the terms involving time:

$-\Gamma^t{}_{rr,t} + \Gamma^t{}_{r\,t,r} + \Gamma^t{}_{\sigma\,r} \Gamma^\sigma{}_{r\,t} - \Gamma^t{}_{\sigma\,t} \Gamma^\sigma{}_{r\,r} + \Gamma^r{}_{t\,r} \Gamma^t{}_{r\,r} - (\Gamma^r{}_{t\,r} + \Gamma^\theta{}_{t\,\theta} + \Gamma^\phi{}_{t\,\phi}) \Gamma^t{}_{rr}$

$= -\Gamma^t{}_{rr,t} + 0 + \Gamma^t{}_{r\,r} \Gamma^r{}_{r\,t} - 0 + \Gamma^r{}_{t\,r} \Gamma^t{}_{r\,r} - (3\,\Gamma^r{}_{t\,r}) \Gamma^t{}_{rr}$

$= -\Gamma^t{}_{rr,t} - \Gamma^t{}_{r\,r} \Gamma^r{}_{r\,t}$

$= -\frac{\partial}{\partial t}\, \dot{a}\, a\, \tilde{g}_{r\,r} - (\dot{a}\, a\, \tilde{g}_{r\,r})\,(\dot{a}\,/a)$

$= -(\ddot{a}\; a + 2\,\dot{a}^2)\, \tilde{g}_{r\,r}$

Then

$$\boxed{R_{rr} = -(\ddot{a}\; a + 2\,\dot{a}^2)\, \tilde{g}_{r\,r} + \tilde{R}_{r\,r}}$$

where

$\tilde{R}_{r\,r} = -\Gamma^i{}_{rr,i} + \Gamma^i{}_{r\,i,r} + \Gamma^i{}_{j\,r} \Gamma^j{}_{r\,i} - \Gamma^i{}_{j\,i} \Gamma^j{}_{rr}$

c) Compute $\tilde{R}_{r\,r}$.

The terms with derivatives:

$-\Gamma^i{}_{rr,i} + \Gamma^i{}_{r\,i,r} = -\Gamma^r{}_{rr,r} + (\Gamma^r{}_{rr,r} + \Gamma^\theta{}_{r\theta,r} + \Gamma^\phi{}_{r\phi,r}) = 2\frac{d}{dr} r^{-1} = -\frac{2}{r^2}$

The other terms:

$\Gamma^i{}_{j\,r} \Gamma^j{}_{r\,i} - \Gamma^i{}_{j\,i} \Gamma^j{}_{rr} = \Gamma^r{}_{rr} \Gamma^r{}_{rr} + \Gamma^\theta{}_{r\theta} \Gamma^\theta{}_{r\theta} + \Gamma^\phi{}_{r\phi} \Gamma^\phi{}_{r\phi} - (\Gamma^r{}_{rr} + \Gamma^\theta{}_{\theta\,r} + \Gamma^\phi{}_{\phi\,r}) \Gamma^r{}_{rr}$

$= 2\frac{1}{r^2} - 2\frac{1}{r}\frac{r}{r_0^2}\frac{1}{1-(r/r_0)^2}$

Finally,

$\tilde{R}_{r\,r} = -\frac{2}{r^2} + \frac{2}{r^2} - \frac{2}{r_0^2}\frac{1}{1-(r/r_0)^2}$

$= -2\, r_0^{-2}\, \tilde{g}_{rr}$

The Christoffel symbol involves first derivatives of the metric tensor. Ricci and Riemann curvature tensors involve second derivatives, and therefore their units are distance^{-2} or time^{-2}.

The space-space part of the Ricci tensor is -2 (radius curvature)$^{-2}$ metric. In n dimensions, the factor 2 is different, but the form is the same.

Since this is true for all of the diagonal element, not only for the r-r term,

$$\boxed{R_{ii} = -(\ddot{a}\; a + 2\,\dot{a}^2 + 2\, r_0^{-2})\, \tilde{g}_{ii}}$$

Friedman's equation

Put Ricci tensor into Einstein's equation to get

$$\boxed{R_{\mu\nu} = -8\,\pi\,G\,S_{\mu\nu}}$$

- **Time-time term**

$$3\,\frac{\ddot{a}}{a} = -8\,\pi\,G\,\frac{1}{2}\,(\rho + 3\,P)$$

$$\ddot{a} = -\frac{4\,\pi}{3}\,G\,(\rho + 3\,P)\,a$$

Q: Interpret the result for the time-time part of Einstein's field equation applied to the R-W metric. Hint: $a = a^3\,a^{-2}$.

- **The i-i term**

$$-\left(\ddot{a}\,a + 2\,\dot{a}^2 + \frac{2}{r_0^2}\right)\tilde{g}_{ii} = -8\,\pi\,G\,\frac{1}{2}\,(\rho - P)\,a^2\,\tilde{g}_{ii}$$

Eliminate $\ddot{a}$ to get

$$\dot{a}^2 - \frac{8\,\pi}{3}\,G\,\rho\,a^2 + \frac{1}{r_0^2} = 0$$

Q: Interpret the result.

Q: What is the big surprise?

5 April 2012—Gravitational waves

- Outline
  - Introduction (§16) (Today)
    How to detect gravity waves (Today)
    Order-of-magnitude strains
    Polarization
  - Wave equation (§21.5)
  - Source of gravitational waves (§23)

Metric for gravitational waves

For a particular gravitational wave, the metric is

$$ds^2 = -dt^2 + [1 + f(t-z)]\,dx^2 + [1 - f(t-z)]\,dy^2 + dz^2$$

where $f(t-z) \ll 1$.

Q: In what way is there a wave?

Q: In which direction is the wave moving?

Q: What is the speed of the wave?

Q: How does the wave affect distances?

How to detect gravitational waves

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$$ds^2 = -dt^2 + [1 + f(t-z)]\,dx^2 + [1 - f(t-z)]\,dy^2 + dz^2$$

where $f(t-z) \ll 1$.

The equation of motion of a mass is

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma}\,u^\beta\,u^\gamma = 0.$$

Since the mass moves slowly, the force term depends only on $u^t = 1$. Then

$$\frac{du^i}{dt} + \Gamma^i{}_{00} = 0.$$

$$\Gamma^x{}_{00} = \frac{1}{2}\,g^{xx}\left(2\,\frac{\partial}{\partial t}\,g_{x\,0} - \frac{\partial}{\partial x}\,g_{00}\right) = 0$$

Surprise: coordinates of the mass do not change.

Q: Simplicio says, "Since the coordinates of every part of my gravity wave detector does not change, there is no way I can detector gravity waves." Is Simplicio correct?

How to detect gravitational waves

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Gravity wave detector: Hold two masses loosely so that there are no other forces. I do not want $\frac{du^i}{dt}$ + forces of bolts = 0

Q: Suppose I want to detect kHz gravity waves. How do I support my gravity wave detector? Write requirements for the supports.

Measure the distance between two parts of the gravity wave detector. The distance is measured with a Michelson interferometer. The interferometer compares the distance between two perpendicular arms. For the Laser Interferometer Gravitational Wave Observatory (LIGO), the arms are 4 km in length. LIGO can detect $f = 10^{-21}$.

Q: What is the detectable change in distance for $L = 4$ km and $f = 10^{-21}$?

$$\delta \mathrm{L} / \lambda = 10^{-11}$$

10^{-21} 4000. Meter / (500*^-9 Meter)

$8. \times 10^{-12}$

Q: Simplicio says, "I want to use very light mirrors for my Michelson interferometer, because for heavy mirrors, the inertia will lessen the response of the interometer to gravity waves." Is Simplicio correct?

Emission of waves. Order-of-magnitude examples

- **Quadrupole oscillations**

A mass M of size a oscillates at angular frequency ω. The system is at distance r. Then the amplitude of the gravitational wave

$$h = 8 \frac{G M a^2 \omega^2}{c^4 r} \text{ (normal units)}$$

$$= 8 \frac{M a^2 \omega^2}{c^2 r} \text{ (mass as a length)}$$

More precisely, the quadrupole moment is $M a^2$.

1. The earth and sun.

Q: What frequency of gravity waves will the earth and sun emit?

Q: What kind of system will produce gravity waves having a detectable frequency?

-

2. Two neutron stars of a solar mass (M=1.5km) separated by $a = 100$ km. (The radius of the sun is 700 Mm. The radius of a neutron star is 10 km.)

Kepler's 3rd law:

$$P^2 = \frac{M_\odot}{M} R^3$$

```
(100. Kilo Meter / Convert[AstronomicalUnit, Kilo Meter])^(3/2) Convert[Year, Second]
```

```
0.0172353 Second
```

The strain at 1pc is (The distance to the nearest star is about 1pc.)

$$h = 10^{-16}.$$

```
Convert[8 × 1.5 Kilo Meter (100 Kilo Meter)^2 1 / %^2 / (SpeedOfLight)^2 / Parsec, 1]
```

```
1.45663 × 10^-16
```

```
Convert[SolarRadius, Mega Meter]
```

```
695.99 Mega Meter
```

Polarization

For the + polarizarion, the metric is

$$ds^2 = -dt^2 + [1 + f(t-z)]\, dx^2 + [1 - f(t-z)]\, dy^2 + dz^2$$

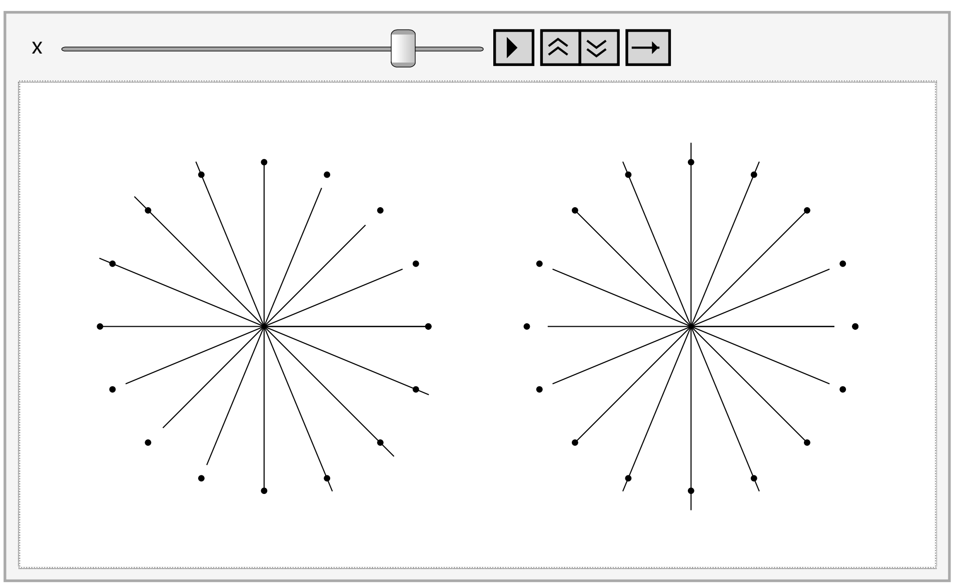
For the × polarization the metric is

$$ds^2 = -dt^2 + dx^2 + f(t-z)\, dx\, dy + dy^2 + dz^2$$

A gravity wave is moving perpendicular to the screen The points are at fixed position. The lines represent the distance between the center and the points.

Q: Which polarization is on the left?

Q: Is an EM wave with y polarization have the same polarization as one of these gravity waves?



Q: A gravity wave moves parallel to an arm of a LIGO interferometer. Is this detectable regardless of polarization?

Q: Most astronomical objects emit unpolarized light. Are gravity waves likely to be unpolarized?

Q: Does the Earth attenuate gravity waves? Are gravity waves weaker if they have to pass through the earth?

- **Plot**

10 Apr 2012—Wave equation