5 April 2012—Gravitational waves

10 Apr 2012—Wave equation for weak gravity waves

- Hwk problem 4. The answer is $d\vec{A} = \left(\frac{\vec{a}\times\vec{b}}{r_0^2}\right)\times\vec{A}$
- Outline
  - Introduction (§16)
    How to detect gravity waves
    Order-of-magnitude strains
    Polarization
  - Wave equation (§21.5) (Today)
  - Source of gravitational waves (§23)

For gravitational waves, the perturbation of the metric is small. Let the metric be

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is small.

Plan:

1) Compute the Christoffel symbols and then the Ricci tensor. Keep first-order terms in h. Then use Einstein's equation

$$R_{\mu\nu} = -8\,\pi\,G\,S_{\mu\nu} = -8\,\pi\,G\left(T_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}\,T\right).$$

to get the wave equation.

2) Solve the wave equation for plane waves.

Einstein's equation for weak fields

The Christoffel symbols

$$\Gamma^{\sigma}{}_{\lambda\mu} = \tfrac{1}{2}\,g^{\nu\sigma}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)$$

The terms $g_{\mu\nu,\lambda}$are first order in h. Therefore we neglect h in the term $g^{\nu\sigma}$.

$$\Gamma^{\sigma}{}_{\lambda\mu} = \tfrac{1}{2}\,\eta^{\nu\sigma}\left(h_{\mu\nu,\lambda} + h_{\lambda\nu,\mu} - h_{\mu\lambda,\nu}\right)$$

The Ricci tensor

$$R_{\mu\kappa} = \frac{\partial}{\partial x^{\kappa}}\,\Gamma^{\lambda}{}_{\mu\lambda} - \frac{\partial}{\partial x^{\lambda}}\,\Gamma^{\lambda}{}_{\mu\kappa} + \Gamma^{\eta}{}_{\mu\lambda}\,\Gamma^{\lambda}{}_{\kappa\eta} - \Gamma^{\eta}{}_{\mu\kappa}\,\Gamma^{\lambda}{}_{\lambda\eta}$$

Since the 3rd and 4th terms are 2$^{\text{nd}}$ order in h, we neglect them.

$$R_{\mu\kappa} = \frac{\partial}{\partial x^{\kappa}}\,\Gamma^{\lambda}{}_{\mu\lambda} - \frac{\partial}{\partial x^{\lambda}}\,\Gamma^{\lambda}{}_{\mu\kappa}$$

Now do the work.

Term $\frac{\partial}{\partial x^{\kappa}}\,\Gamma^{\lambda}{}_{\mu\lambda}$:

$$\Gamma^{\lambda}{}_{\lambda\mu} = \tfrac{1}{2}\,\eta^{\nu\lambda}\left(h_{\mu\nu,\lambda} + h_{\lambda\nu,\mu} - h_{\mu\lambda,\nu}\right)$$

Since I can swap ν and λ on RHS, 1st and 3rd terms cancel.

$$\frac{\partial}{\partial x^{\kappa}}\,\Gamma^{\lambda}{}_{\mu\lambda} = \frac{1}{2}\,\eta^{\nu\lambda}\,\frac{\partial^2}{\partial x^{\mu}\,\partial x^{\kappa}}\,h_{\lambda\nu} = \frac{1}{2}\,\frac{\partial^2}{\partial x^{\mu}\,\partial x^{\kappa}}\,h^{\lambda}_{\lambda}$$

The other term

$$-\frac{\partial}{\partial x^{\lambda}}\,\Gamma^{\lambda}{}_{\mu\kappa} = -\frac{1}{2}\,\eta^{\nu\lambda}\left(h_{\mu\nu,\kappa,\lambda} + h_{\kappa\nu,\mu,\lambda} - h_{\mu\kappa,\nu,\lambda}\right)$$
$$= -\frac{1}{2}\left(h^{\lambda}_{\mu,\kappa,\lambda} + h^{\lambda}_{\kappa,\mu,\lambda} - h_{\mu\kappa,\lambda}^{,\lambda}\right)$$

Notice the derivative on the upper index. What does $h_{\mu\kappa,\lambda}^{,\lambda}$ mean? It is $\eta^{\nu\lambda}\,h_{\mu\kappa,\nu,\lambda}$

Since η is diagonal, $\nu = \lambda$ and

$$\eta^{\gamma\lambda}\, h_{\mu\kappa,\nu,\lambda} = \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) h_{\mu\nu} = \Box\, h_{\mu\kappa}$$

This is the d'Alembertian.

The whole works:

$$R_{\mu\kappa} = -\frac{1}{2}\left(\frac{\partial}{\partial x^\kappa} A_\mu + \frac{\partial}{\partial x^\mu} A_\kappa - h_{\mu\kappa,\lambda}^{,\lambda}\right),$$

where

$$A_\mu = \frac{\partial}{\partial x^\lambda} h_\mu^\lambda - \frac{1}{2}\frac{\partial}{\partial x^\mu} h_\lambda^\lambda.$$

Finally, Einstein's field equation is

$$\boxed{R_{\mu\kappa} = -\frac{1}{2}\left(\frac{\partial}{\partial x^\kappa} A_\mu + \frac{\partial}{\partial x^\mu} A_\kappa - \Box\, h_{\mu\kappa}\right) = -8\,\pi\, G\, S_{\mu\nu}}$$

Q: What do the terms mean? Is this a customary wave equation?

Unphysical freedom

There is freedom in h. Change the coordinate system. Replace x^α by

$$x'^\alpha = x^\alpha + \epsilon^\alpha(x),$$

where $\epsilon^\alpha(x)$ is small just like $h_{\alpha\beta}$. Then

$$\eta^{\alpha\beta} + h'^{\,\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\left(\eta^{\mu\nu} + h^{\mu\nu}\right)$$

$$= \left(\delta_\mu^\alpha + \frac{\partial \epsilon^\alpha}{\partial x^\mu}\right)\left(\delta_\nu^\beta + \frac{\partial \epsilon^\beta}{\partial x^\nu}\right)\left(\eta^{\mu\nu} + h^{\mu\nu}\right)$$

$$= \eta^{\alpha\beta} + h^{\alpha\beta} + \frac{\partial \epsilon^\alpha}{\partial x^\mu}\,\eta^{\mu\beta} + \frac{\partial \epsilon^\beta}{\partial x^\nu}\,\eta^{\alpha\nu}$$

I ignored the 2nd-order terms to write the last line. The 1st order term is

$$h'^{\alpha\beta} = h^{\alpha\beta} + \frac{\partial \epsilon^\alpha}{\partial x^\mu}\,\eta^{\mu\beta} + \frac{\partial \epsilon^\beta}{\partial x^\nu}\,\eta^{\alpha\nu}$$

Upon a change in the coordinate system, the strain changes to

$$\boxed{h'_{\alpha\beta} = h_{\alpha\beta} + \frac{\partial \epsilon_\alpha}{\partial x^\beta} + \frac{\partial \epsilon_\beta}{\partial x^\alpha}}$$

Choose a "gauge" so that

$$\boxed{A_\mu = \frac{\partial}{\partial x^\lambda} h_\mu^\lambda - \frac{1}{2}\frac{\partial}{\partial x^\mu} h_\lambda^\lambda = 0}.$$

If h does not satisfy this condition, then choose a transformation $\epsilon(x)$ so that it does.
Then we have the wave equation

$$\boxed{\Box h_{\mu\kappa} \equiv \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) h_{\mu\kappa} = -16\,\pi\, G\, S_{\mu\kappa}}.$$

Solution of the wave equation

Guess a plane-wave solution

$$h_{\alpha\beta}(x^\gamma) = a_{\alpha\beta}\, e^{i\, k_\gamma\, x^\gamma}.$$

Q: How do you take derivatives of h? Compute $\frac{\partial^2}{\partial t^2} h_{\alpha\beta}$.

For this to satisfy the wave equation,

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) a_{\alpha\beta}\, e^{i\, k_\gamma\, x^\gamma} = a_{\alpha\beta}\, e^{i\, k_\gamma\, x^\gamma}\left(-k_0^2 + k_x^2 + k_y^2 + k_z^2\right) = 0,$$

or for some $a_{\alpha\beta} \neq 0$,

$$k_\mu k^\mu = 0.$$

Q: Interpret $k_\mu k^\mu = 0$.

For the solution to satisy the gauge,

$$k_\lambda a^\lambda_\beta e^{\iota k_\gamma x^\gamma} = \frac{1}{2} k_\beta a^\lambda_\lambda e^{\iota k_\gamma x^\gamma}.$$

$$k_\lambda a^\lambda_\beta = \frac{1}{2} k_\beta a^\lambda_\lambda.$$

Problem: Consider a wave traveling in the z direction.

$$k^\mu = (k, 0, 0, k),$$

where $k > 0$. Determine the possible values for $a_{\alpha\beta}$.

The gauge condition $k^\lambda a_{\lambda\beta} = \frac{1}{2} k_\beta a^\lambda_\lambda$.

$$k a_{0\beta} + k a_{3\beta} = \frac{1}{2} k_\beta(-a_{00} + a_{11} + a_{22} + a_{33})$$

gives

$$a_{00} + a_{30} = -\frac{1}{2} (-a_{00} + a_{11} + a_{22} + a_{33})$$

$$a_{01} + a_{31} = a_{02} + a_{32} = 0$$

$$a_{03} + a_{33} = \frac{1}{2} (-a_{00} + a_{11} + a_{22} + a_{33})$$

So

$$a_{01} = -a_{31}$$

$$a_{02} = -a_{32}$$

$$a_{03} = -\frac{1}{2} (a_{33} + a_{00})$$

$$a_{22} = -a_{11}$$

Replace x^α by

$$x'^\alpha = x^\alpha + \epsilon^\alpha(x).$$

Then $h'_{\alpha\beta} = h_{\alpha\beta} + \frac{\partial \epsilon_\alpha}{\partial x^\beta} + \frac{\partial \epsilon_\beta}{\partial x^\alpha}$ yields

$$a'_{11} = a_{11} + 2 k_1 \epsilon_1 = a_{11}$$

$$a'_{12} = a_{12} + k_1 \epsilon_2 + k_2 \epsilon_1 = a_{12}$$

$$a'_{13} = a_{13} + k_1 \epsilon_3 + k_3 \epsilon_1 = a_{13} + k \epsilon_1$$

$$a'_{23} = a_{23} + k_2 \epsilon_3 + k_3 \epsilon_2 = a_{23} + k \epsilon_2$$

$$a'_{33} = a_{33} + 2 k_3 \epsilon_3 = a_{33} + 2 k \epsilon_3$$

$$a'_{00} = a_{00} - 2 k_0 \epsilon_0 = a_{00} - 2 k \epsilon_0$$

because $k_1 = k_2 = 0$.

Q: Interpret these equations. Which $a_{\alpha\beta}$ are physical?

- **Physical polarizations**

I can choose ϵ_1 to eliminate a_{13}. Similarly I can eliminate a_{23}, a_{33}, and a_{00}.
Setting $a_{33} = 0$ removes the longitudinal polarization.

a_{11} and a_{12} do not change. They cannot be removed with a coordinate change.

There are two independent numbers a_{11} and a_{12}. The two independent polarizations are

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- **Resume**

Q: Simplicio: If I did not choose to eliminate a_{33}, then there would be additional terms in the metric
$ds^2 = -dt^2 + [1 + f(t-z)]\,dx^2 + [1 - f(t-z)]\,dy^2.$
$+[1 + w(t-z)]\,dz^2 - w(t-z)\,dt\,dz$
I could make $w(t-z)$ really big and detect gravity waves easily. What is wrong?

- **Ans**

Spin of gravity waves

Consider the polarization

$$a_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rotate by angle θ about the z axis. The transformation is

$\Lambda_1{}^1 = \cos\theta$

$\Lambda_1{}^2 = \sin\theta$

$\Lambda_2{}^1 = -\sin\theta$

$\Lambda_2{}^2 = \cos\theta$

$\Lambda_0{}^0 = \Lambda_3{}^3 = 1$

and the other terms are 0.

$a'_{\alpha\beta} = \Lambda_\alpha{}^\gamma \Lambda_\beta{}^\delta a_{\gamma\delta}$

Do the 11 term

$a'_{11} = \Lambda_1{}^\gamma \Lambda_1{}^\delta a_{\gamma\delta} = \Lambda_1{}^1 \Lambda_1{}^1 a_{11} + \Lambda_1{}^2 \Lambda_1{}^1 a_{21} + \Lambda_1{}^1 \Lambda_1{}^2 a_{12} + \Lambda_1{}^2 \Lambda_1{}^2 a_{22}$

$= \cos^2\theta - 2\,i\sin\theta\cos\theta - \sin^2\theta$

$= \cos 2\theta - i\sin 2\theta$

$= e^{2 i\theta}$

We can show that

$a'_{\alpha\beta} = e^{2 i\theta} a_{\alpha\beta}$

For the polarization

$$b_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$b'_{\alpha\beta} = e^{-2 i\theta} b_{\alpha\beta}$

Q: For what rotation angle is the rotated wave the same as the original wave?

Gravity waves are spin 2.