5 April 2012—Gravitational waves

10 Apr 2012—Wave equation for weak gravity waves

12 Apr 2012 Radiation from a source

- Outline
  - Introduction (§16)
    How to detect gravity waves
    Order-of-magnitude strains
    Polarization
  - Wave equation (§21.5)
  - Source of gravitational waves (§23) (Today)
  - The Hulse-Taylor pulsar

Source of gravitational radiation

The wave equation

$$\boxed{\Box\, h_{\mu\kappa} \equiv \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) h_{\mu\kappa} = -16\,\pi\, G\, S_{\mu\kappa}}.$$

In E&M, you learned the solution to the wave equation. This is derived in Hartle (§23.2)

$$h_{\alpha\beta}(t, \vec{x}) = 4\, G \int \frac{S_{\alpha\beta}(t', \vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3 x'$$

Recall the source $S_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, T^{\lambda}_{\lambda}$.

Is this plausible?

For static masses, what should $h_{\alpha\beta}$be? Recall the Schwarzschild metric in homogeneous coordinates.

$$g_{\mu\nu} = \begin{pmatrix} -(1 - 2\,M/r) & 0 & 0 & 0 \\ 0 & (1 - 2\,M/r)^{-1} & 0 & 0 \\ 0 & 0 & (1 - 2\,M/r)^{-1} & 0 \\ 0 & 0 & 0 & (1 - 2\,M/r)^{-1} \end{pmatrix}$$

For $M/r \ll 1$, $-M/r$ is the Newtonian gravitational potential.

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} = \begin{pmatrix} 2\,\phi & 0 & 0 & 0 \\ 0 & -2\,\phi & 0 & 0 \\ 0 & 0 & -2\,\phi & 0 \\ 0 & 0 & 0 & -2\,\phi \end{pmatrix}$$

What does the formula give? For static masses, $T_{\alpha\beta} = \text{diag}(\rho, 0, 0, 0)$, and $T^{\lambda}_{\lambda} = -\rho$. So $S_{\alpha\beta} = \frac{1}{2}\,\text{diag}(\rho, -\rho, -\rho, -\rho)$.

$$h_{\alpha\beta} = 4 \int S_{\alpha\beta} \big/ r\, \text{dVol} = -\frac{2\,M}{r}\,\text{diag}(1, -1, -1, -1) = 2\,\text{diag}(\phi, -\phi, -\phi, -\phi).$$

There is a time t' in the integral. How do you define t'?

The field at $(t, \vec{x})$ depends on the source at $(t', \vec{x}')$. The effect of the source at t' must travel at the speed of light.

$$t' = t - \left|\vec{x} - \vec{x}'\right|$$

The quadrupole formula

Consider a slowly moving source that is confined to a region much smaller than the distance to the observer. Then

$$h_{\alpha\beta}(t, \vec{x}) = 4\,G \int \frac{S_{\alpha\beta}(t', \vec{x}')}{\left|\vec{x}-\vec{x}'\right|}\, d^3 x'$$

$$= 4\,G\,\frac{1}{r}\int S_{\alpha\beta}(t', \vec{x'})\, d^3 x'$$

We found that the trace $h_\alpha{}^\alpha$ is zero. The two polarizations were $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then we can use T rather than S and follow the derivation in Hartle 23.4.

$$h^{\alpha\beta}(t, \vec{x}) = 4\,G\,\frac{1}{r}\int T^{\alpha\beta}(t', \vec{x'})\, d^3 x'.$$

The problem is that the masses in the source can move. Using the conservation law $\nabla T^{\alpha\beta} = 0$, we will find that

$$h^{ij}(t, r) = G\,\frac{1}{r}\,\frac{d^2}{dt^2}\,I^{ij}(t),$$

where the

$$I^{ij} = \int x^i\, x^j\, \rho\,(t, \vec{x})\, d^3 x.$$

Integrate by parts to change a volume integral into a surface integral. (Supress the primes, since we know we are at the source and not the observer.)

$$\int T^{ij}(t, \vec{x})\, d^3 x = -\int x^i\,\frac{\partial T^{ij}}{\partial x^i}\, d^3 x + x^i\, T^{ij} \text{ on surface perpendicular to i}$$

Away from the source, $T^{ij} = 0$. The surface term is zero. Do the same trick once again.

$$\int T^{ij}(t, \vec{x})\, d^3 x = \int x^i\, x^j\,\frac{\partial^2 T^{ij}}{\partial x^i\,\partial x^j}\, d^3 x$$

Use the conservation of energy and momentum

$$\frac{\partial T^{tt}}{\partial t} + \frac{\partial T^{it}}{\partial x^i} = 0 \text{ (sum over i)}$$

and once again

$$\frac{\partial^2 T^{tt}}{\partial t^2} = -\frac{\partial^2 T^{it}}{\partial x^i\,\partial t} = \frac{\partial^2 T^{ij}}{\partial x^i\,\partial x^j}$$

$$\int T^{ij}(t, \vec{x})\, d^3 x = \frac{\partial^2}{\partial t^2}\int x^i\, x^j\, T^{tt}\, d^3 x$$

Since $T^{ii} = \rho$,
The quantity

$$I^{ij} \equiv \int x^i\, x^j\, \rho(t, x)\, d^3 x$$

is called the second moment of mass. The moment of inertia is one term for a symmetric mass distribution.

We have found

$$\boxed{h^{ij}(t, r) = G\,\frac{1}{r}\,\frac{d^2}{dt^2}\,I^{ij}}$$

Q: Two stars of equal mass orbit each other. Does this system emit gravitational radiation perpendicular to the plane of the orbit? Does this system emit gravitational radiation in the plane of the orbit?

Q: Two stars of equal mass orbit each other. If the period is half, how much stronger is the radiation? Assume the masses are the same. The orbital radius must be smaller by a factor $2^{2/3}$ because of Kepler's 3rd law $P^2 = R^3$.
A: $2^{2/3}$

Binary pulsar

For a binary star system, the energy radiated in gravitational waves causes the period to change (Taylor & Weisberg)

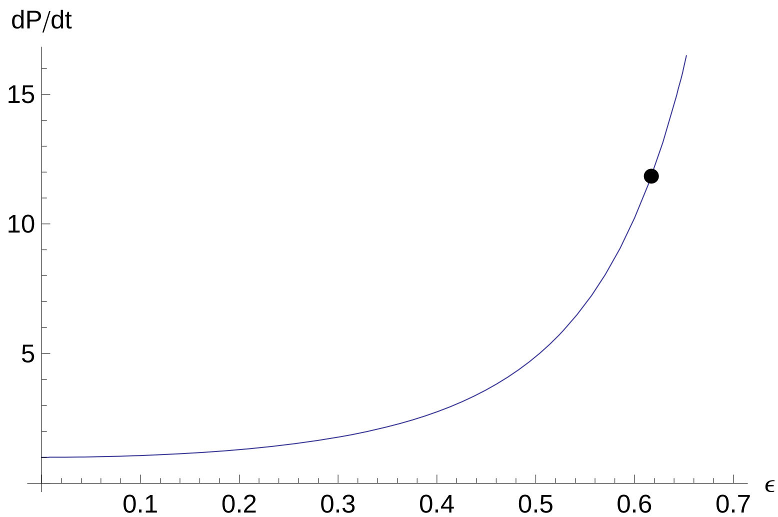
$$\dot{P} \equiv \frac{dP}{dt} = -\frac{192}{5}\left(\frac{2\pi}{P}\right)^{5/3} m_1\, m_2\, M^{-1/3}\, f(\epsilon)$$

The masses of the stars are $m_1$and m_2. $M = m_1 + m_2$. The eccentricity is ϵ.

$$f(\epsilon) = \left(1 + \frac{73}{24}\epsilon^2 + \frac{37}{96}\epsilon^4\right)\left(1-\epsilon^2\right)^{-7/2}$$

The behavior $\dot{P} \propto P^{-5/3}$ will be discussed the next class.

Q: Simplicio: If a system loses energy, it should slow down, not speed up. What is wrong with Simplicio's thinking when applied to the binary pulsar?

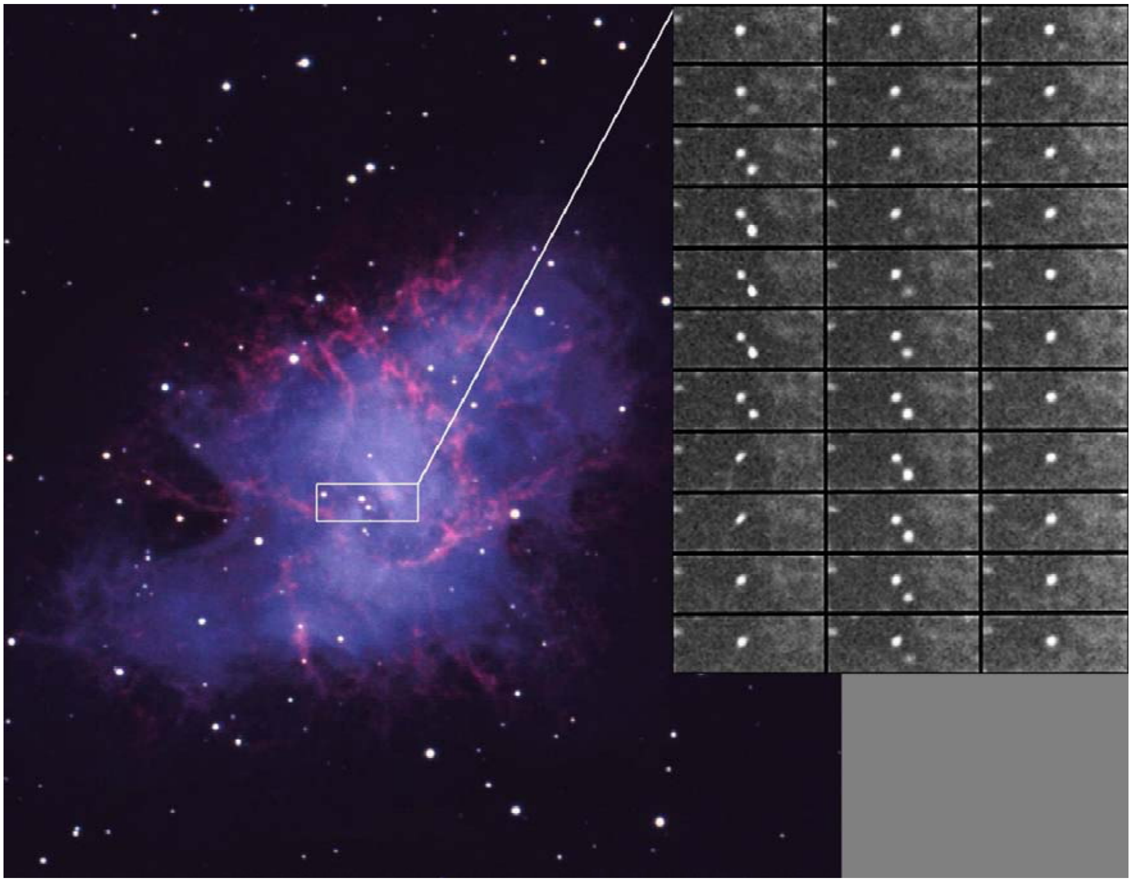


Q: Why is the period derivative (and gravitational radiation) large for high eccentricity?

- **Plot**

Pulsars

Pulsars are spinning neutron stars.
A neutron star is made of neutrons. If the density is high enough, a neutron has a lower energy than a proton and an electron.
The size of neutron star is 20km.
Radiation comes from the magnetic poles, which points towards Earth at certain times in its period.
Crab pulsar. The period of the Crab Pulsar is 33ms.

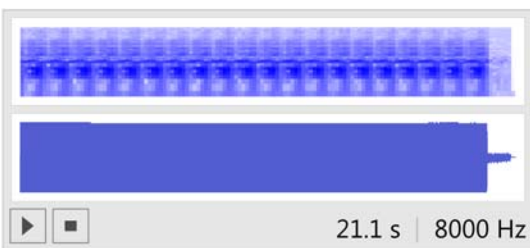


http://www.jb.man.ac.uk/~pulsar/Education/Sounds/sounds.html

In[3770]:=
```
crabSound = Import["c:\\users\\loh\\desktop\\crab.au"];
```

In[3771]:=
```
crabSound
```

Out[3771]=



Binary pulsar 1913+16

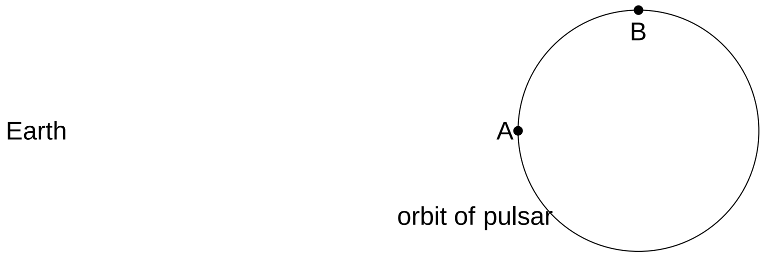
Observations of the binary pulsar 1913+16 was the first detection of the effect of gravitational waves.
The pulsar emits a burst of radio waves every 59ms.
Because of gravitational radiation, the system is losing energy, and the orbital period becomes shorter.

Ref: Taylor & Weisberg 1989, ApJ, 345, 434.
Taylor 1994, RMP 66.
Hulse 1994, RMP 66.

■ Pulsars are precise probes of space and time

Imagine a system of two neutron stars. One is a pulsar.

Out[3769]=



Q: How would an observer on Earth know whether the pulsar is at A or B?

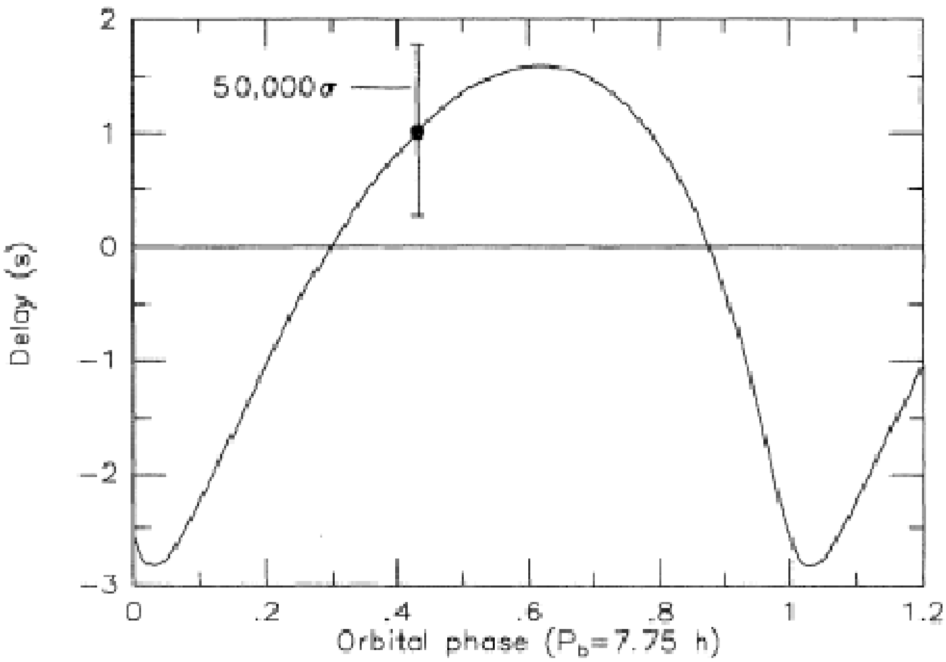


FIG. 5. Orbital delays observed for PSR 1913+16 during July, 1988. The uncertainty of an individual five-minute measurement is typically 50 000 times smaller than the error bar shown.

Taylor 1994, RMP

The period of 1913+16 is 27906.980894(2)s (11 digits), about 8hr. The uncertainty in the time of arrival (5-min average) in 1988 was 16μs. How well can the position of the pulsar be determined?

```
16*^-6 × 3.*^5 km
```

```
4.8 km
```

The radius of the orbit is 2.3s

```
a = 2.3 × 3*^5 km
```

```
690 000. km
```

What fraction of the orbit is the positional accuracy? A part in

```
% / %%
```

```
143 750.
```

Q: Why were the effects of gravity waves not seen in the thousands of observations of other stars? What were keys about binary pulsars?

- **Pulsars measure g_{00} and other general relativistic parameters**

The eccentricity is 0.6171472(10). The gravitational redshift is

$$-\frac{M}{r}$$

It changes because r changes. $(1-\epsilon)\,a < r < (1+\epsilon)\,a$

```
ϵ = 0.6171472;
1.5 km * 1.4 / a / {1 - ϵ, 1 + ϵ}
```

$\{7.81359 \times 10^{-6},\ 1.84983 \times 10^{-6}\}$

Q: How would gravitational redshift appear in the observations?

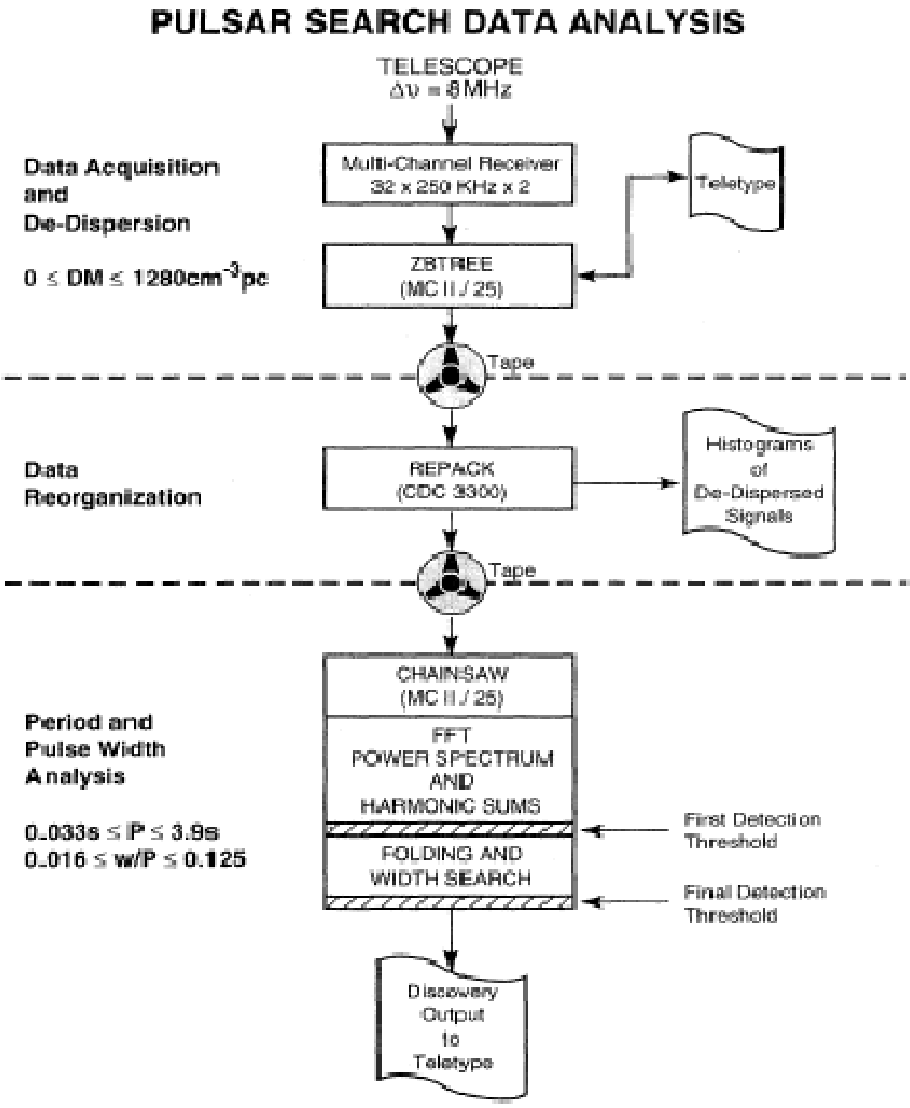
- **Fig**

Discovery of the binary pulsar

Russell Hulse and Joe Taylor in 1974. Figures from Hulse's Nobel paper in RMP

In principle, finding pulsars is straightforward. Look for periodic signals from the sky.
The problem is dispersion by interstellar electrons. The velocity of the radio waves is proportional to $1/f^2$. A "dedisperser" removes the dispesion of the interstellar medium.

The dedisperser was done by Chainsaw.
Guess how much memory the Chainsaw computer had? 32kB, 32MB, 32GB?


PULSAR SEARCH DATA ANALYSIS
TELESCOPE
Δν = 8 MHz
Data Acquisition
and
De-Dispersion
0 ≤ DM ≤ 1280cm⁻³pc
Multi-Channel Receiver
32 x 250 KHz x 2
Teletype
ZBTREE
(MC II / 25)
Tape
Data
Reorganization
REPACK
(CDC 3300)
Histograms
of
De-Dispersed
Signals
Tape
CHAINSAW
(MC II / 25)
Period and
Pulse Width
Analysis
0.033s ≤ P ≤ 3.9s
0.016 ≤ w/P ≤ 0.125
FFT
POWER SPECTRUM
AND
HARMONIC SUMS
First Detection
Threshold
FOLDING AND
WIDTH SEARCH
Final Detection
Threshold
Discovery
Output
to
Teletype

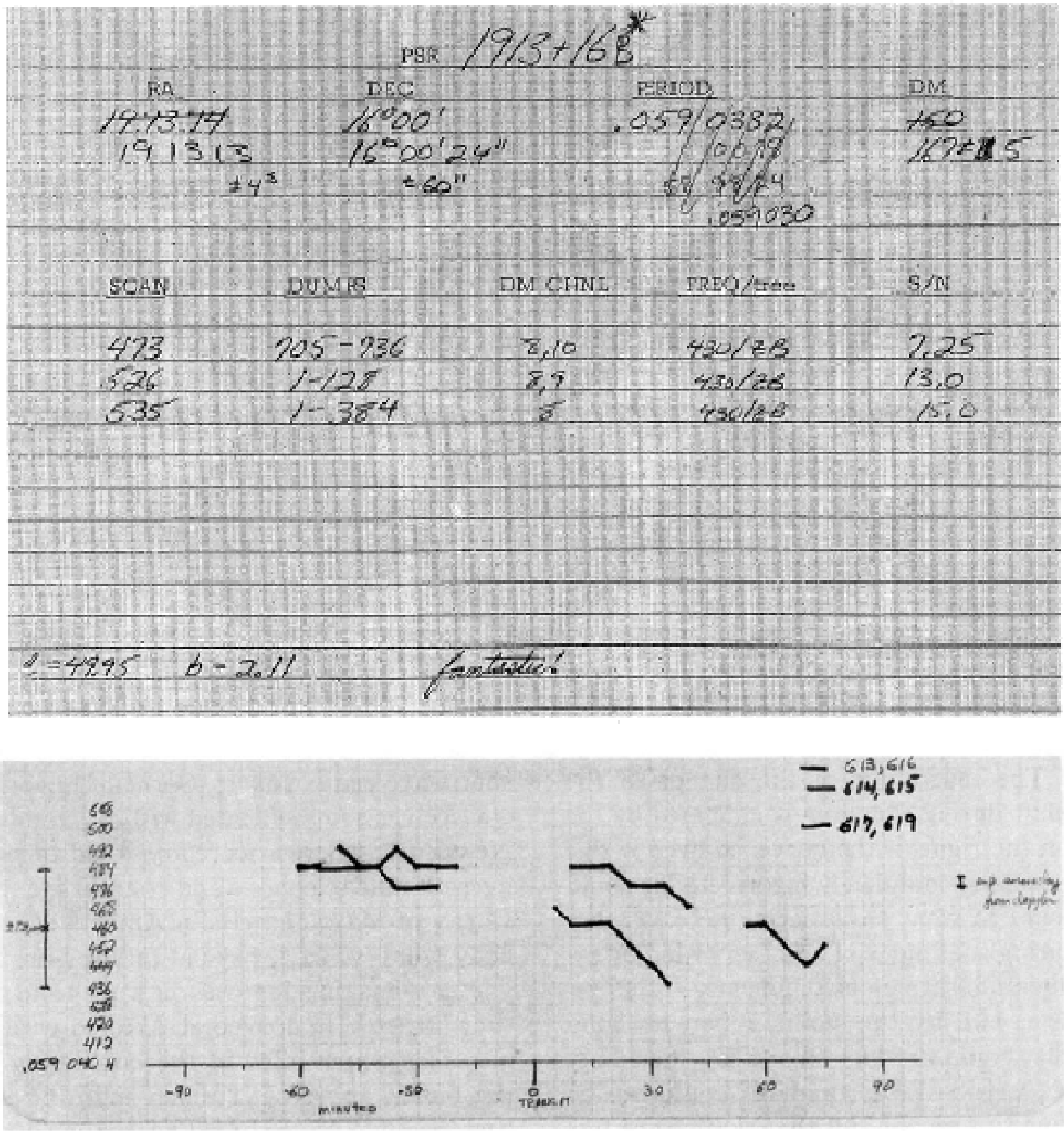


FIG. 9. The first two observations of PSR 1913+16 using the improved system specially designed to resolve the difficulties with determining the pulsation period for this pulsar. The observed pulsation period in successive 5-minute integrations is plotted versus time before and after transit. A calculation showing the magnitude of the change in the earth's Doppler shift is also seen on the right. Looking at this plot of data from September 1 and September 2, I realized that by shifting the second of these curves by 45 minutes the two curves would overlap. This was a key moment in deciphering the binary nature of PSR 1913+16.

—17 Apr 2012