

Chapter 9

Rotational Dynamics

9.1 *The Action of Forces and Torques on Rigid Objects*

Chapter 8 developed the concepts of angular motion.

θ : angles and radian measure for angular variables

ω : angular velocity of rotation (same for entire object)

α : angular acceleration (same for entire object)

$v_T = \omega r$: tangential velocity

$a_T = \alpha r$: tangential acceleration

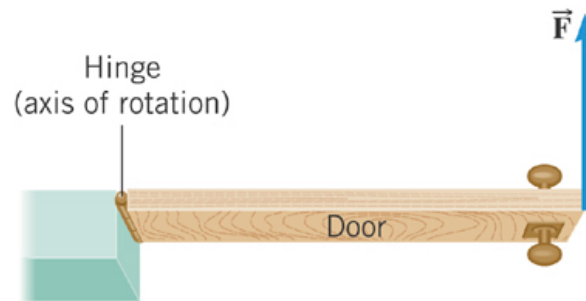
According to Newton's second law, a net force causes an object to have a ***linear acceleration***.

What causes an object to have an ***angular acceleration***?

TORQUE

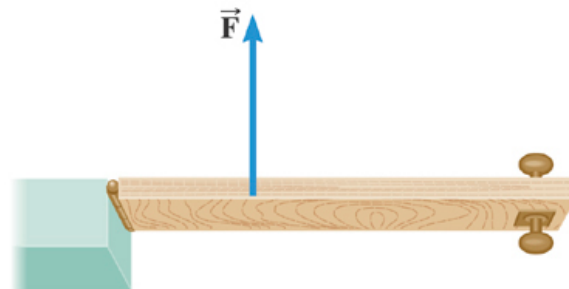
9.1 *The Action of Forces and Torques on Rigid Objects*

The amount of torque depends on where and in what direction the force is applied, as well as the location of the axis of rotation.



Maximum rotational effect of the force F .

(a)



Smaller rotational effect of the force F .

(b)



Rotational effect of the force F is minimal; it compresses more than rotates the bar

(c)

9.1 The Action of Forces and Torques on Rigid Objects

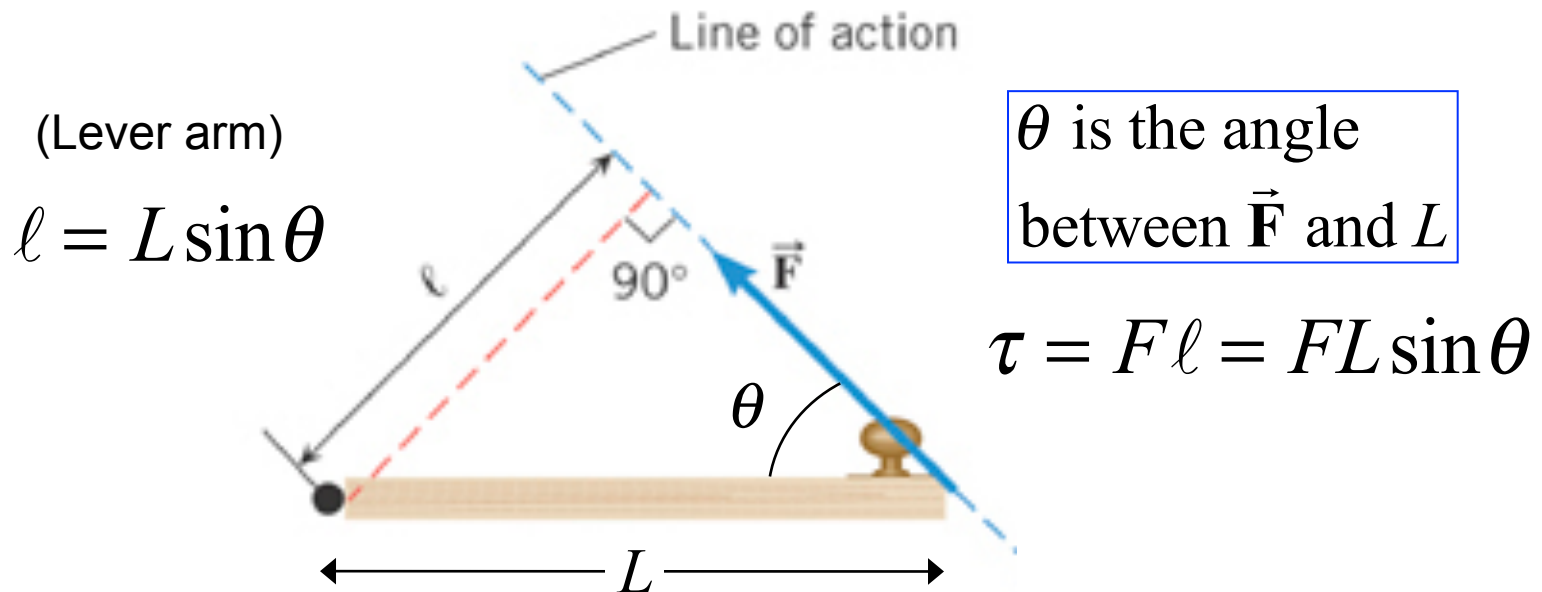
DEFINITION OF TORQUE

Magnitude of Torque = (Magnitude of the force) x (Lever arm)

$$\tau = F\ell$$

Direction: The torque is **positive** when the force tends to **produce a counterclockwise rotation** about the axis.

SI Unit of Torque: newton x meter (N·m)



9.1 The Action of Forces and Torques on Rigid Objects

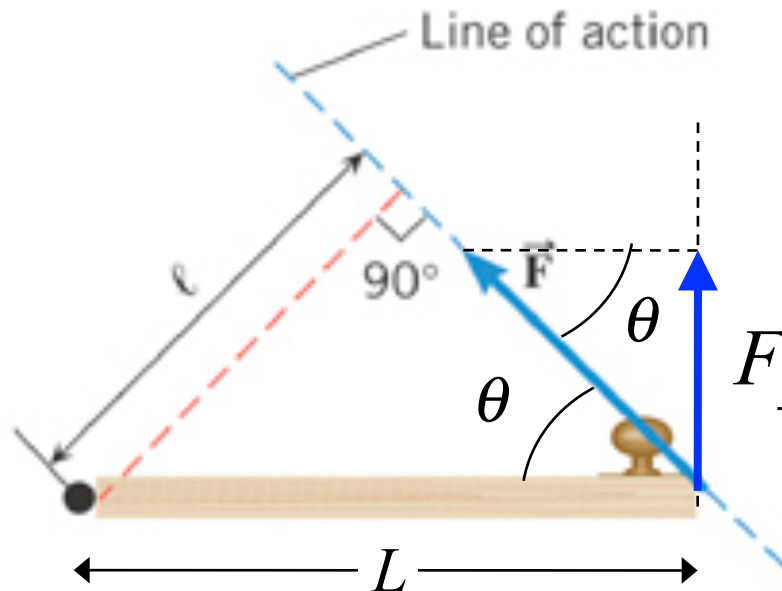
Magnitude of Torque = (Magnitude of the force) $\times$ (Lever arm)

$$\tau = F\ell = FL\sin\theta \quad \ell = L\sin\theta$$

Alternate (Equivalent) Interpretation

Magnitude of Torque = (Component of Force $\perp$ to L) $\times L$

$$\tau = F_{\perp}L = (F\sin\theta)(L) = FL\sin\theta$$



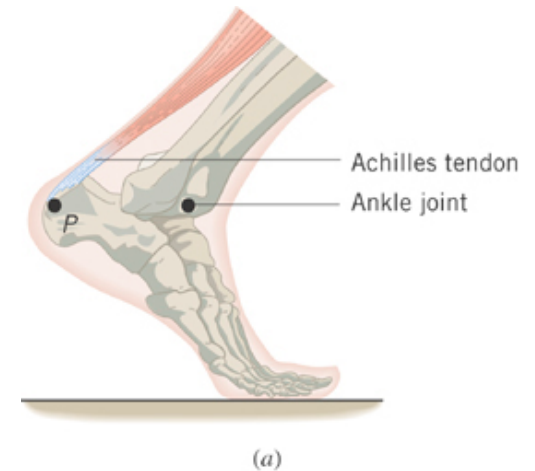
θ is the angle
between $\vec{F}$ and L

$$F_{\perp} = F \sin \theta$$

9.1 The Action of Forces and Torques on Rigid Objects

Example 2 The Achilles Tendon

The tendon exerts a force of magnitude 720 N. Determine the torque (magnitude and direction) of this force about the ankle joint.

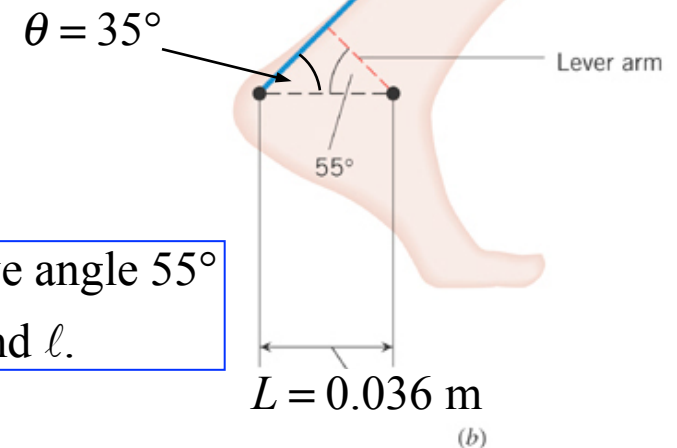


$$\tau = (F \sin \theta) L = (720 \text{ N})(\sin 35^\circ)(.036 \text{ m}) \\ = 15.0 \text{ N} \cdot \text{m}$$

θ is the angle
between $\vec{F}$ and L

$$\tau = F(L \sin \theta) = (720 \text{ N})(.036 \text{ m}) \sin 35^\circ \\ = 15.0 \text{ N} \cdot \text{m}$$

Example gave angle 55°
between L and ℓ .



Direction is clockwise (–) around ankle joint

Torque vector $\tau = -15.0 \text{ N} \cdot \text{m}$

9.2 *Rigid Objects in Equilibrium*

If a rigid body is in equilibrium, neither its linear motion nor its rotational motion changes.

$$a_x = a_y = 0$$

$$\alpha = 0$$

$$\sum F_x = 0 \quad \sum F_y = 0$$

$$\sum \tau = 0$$

All **equilibrium** problems use these equations – no net force and no net torque.

9.2 *Rigid Objects in Equilibrium*

EQUILIBRIUM OF A RIGID BODY

A rigid body is in equilibrium if it has **zero translational acceleration** and **zero angular acceleration**. In equilibrium, the sum of the externally applied forces is zero, and the sum of the externally applied torques is zero.

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum \tau = 0$$

Note: **constant linear speed** or **constant rotational speed** are allowed for an object in equilibrium.

9.2 *Rigid Objects in Equilibrium*

Reasoning Strategy

1. Select the object to which the equations for equilibrium are to be applied.
2. Draw a free-body diagram that shows all of the external forces acting on the object.
3. Choose a convenient set of x , y axes and resolve all forces into components that lie along these axes.
4. Apply the equations that specify the balance of forces at equilibrium. (Set the net force in the x and y directions equal to zero.)
5. Select a convenient axis of rotation. Set the sum of the torques about this axis equal to zero.
6. Solve the equations for the desired unknown quantities.

9.2 Rigid Objects in Equilibrium

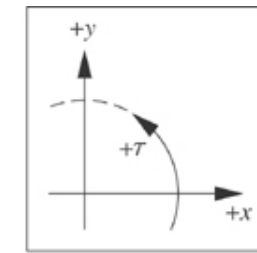
Example 3 A Diving Board

A woman whose weight is 530 N is poised at the right end of a diving board with length 3.90 m. The board has negligible weight and is supported by a fulcrum 1.40 m away from the left end. Find the forces that the bolt and the fulcrum exert on the board.

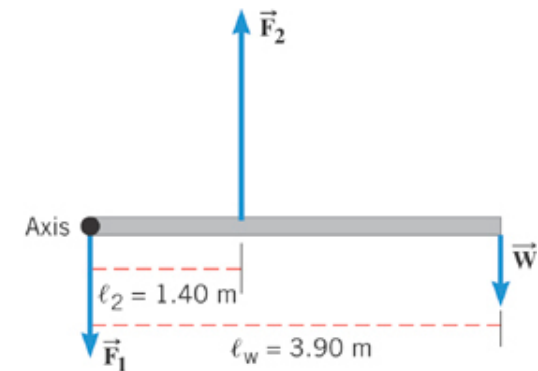
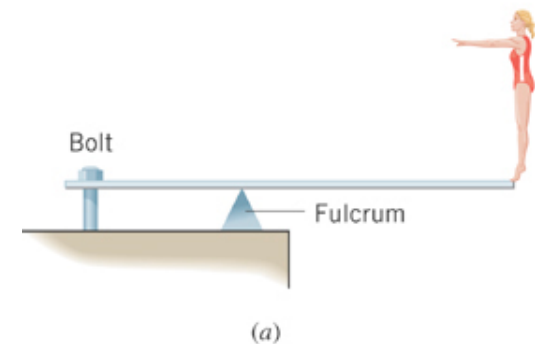
F_1 acts on rotation axis - produces no torque.

$$\sum \tau = 0 = F_2 \ell_2 - W \ell_w$$
$$F_2 = W(\ell_w / \ell_2) = 530 \text{ N}(3.9/1.4) = 1480 \text{ N}$$

$$\sum F_y = 0 = -F_1 + F_2 - W$$
$$F_1 = F_2 - W = (1480 - 530) \text{ N} = 950 \text{ N}$$



Counter-clockwise torque is positive



(b) Free-body diagram of the diving board

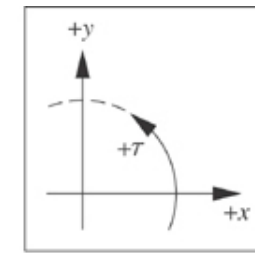
9.2 Rigid Objects in Equilibrium

Choice of pivot is arbitrary (most convenient)

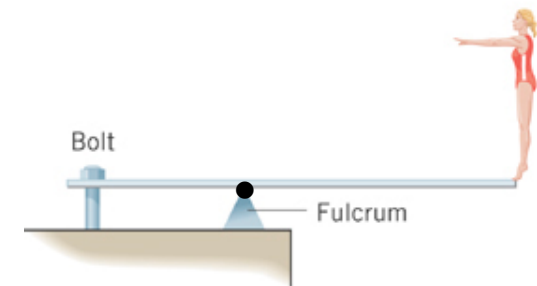
Pivot at fulcrum: F_2 produces no torque.

$$\sum \tau = 0 = F_1 \ell_2 - W(\ell_w - \ell_2)$$
$$F_1 = W(\ell_w / \ell_2 - 1) = (530\text{N})(1.8) = 950\text{ N}$$

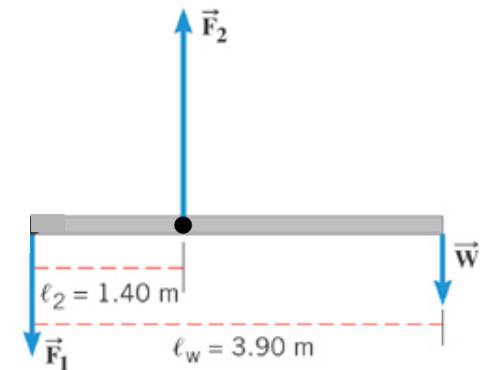
$$\sum F_y = 0 = -F_1 + F_2 - W$$
$$F_2 = F_1 + W = (950 + 530)\text{N} = 1480\text{ N}$$



Counter-clockwise torque is positive



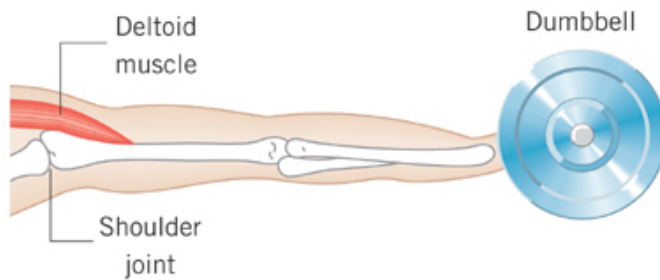
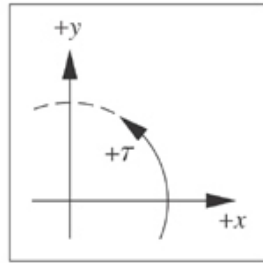
(a)



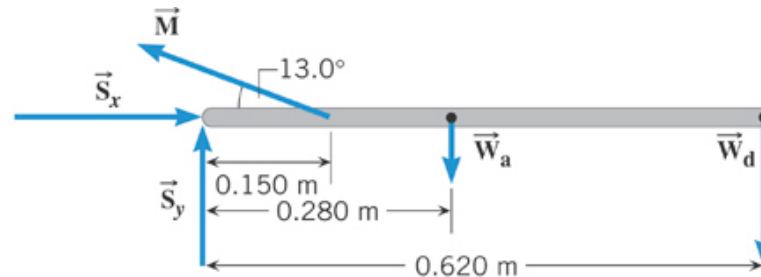
(b) Free-body diagram of the diving board

9.2 Rigid Objects in Equilibrium

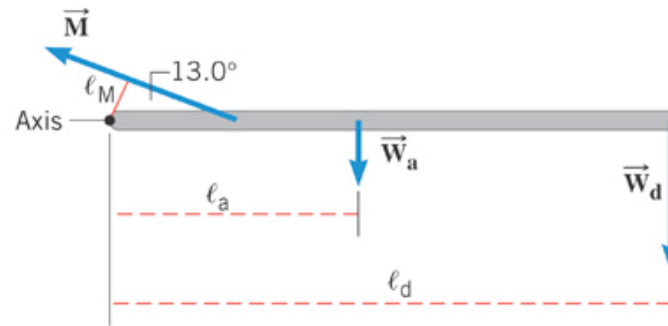
Counter-clockwise
torque is positive



(a)



(b) Free-body diagram of the arm

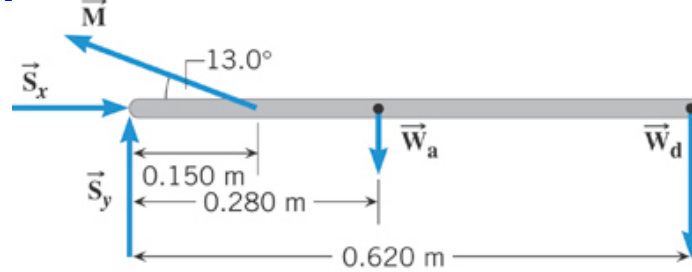


(c)

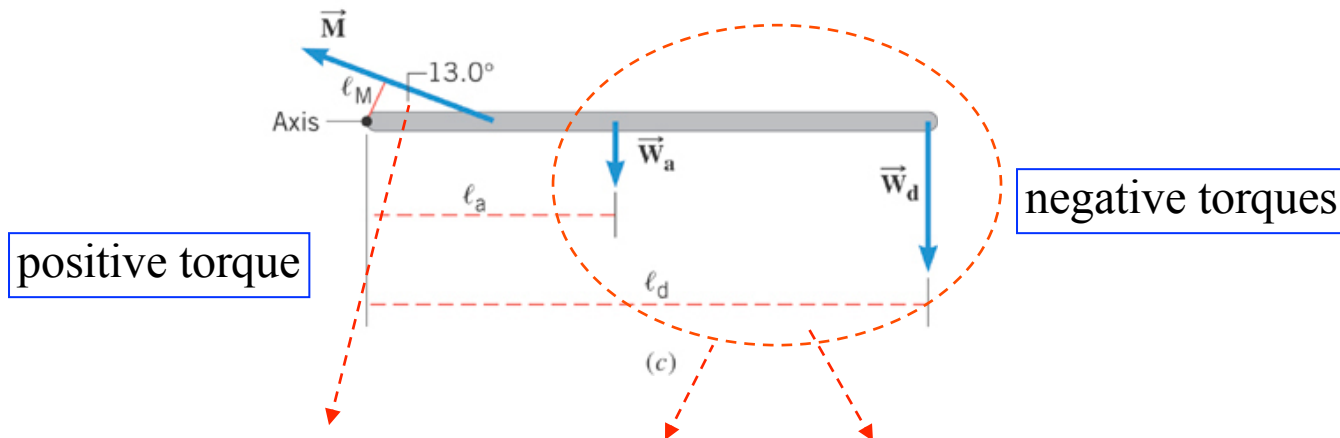
Example 5 Bodybuilding

The arm is horizontal and weighs 31.0 N. The deltoid muscle can supply 1840 N of force. What is the weight of the heaviest dumbbell he can hold?

9.2 Rigid Objects in Equilibrium



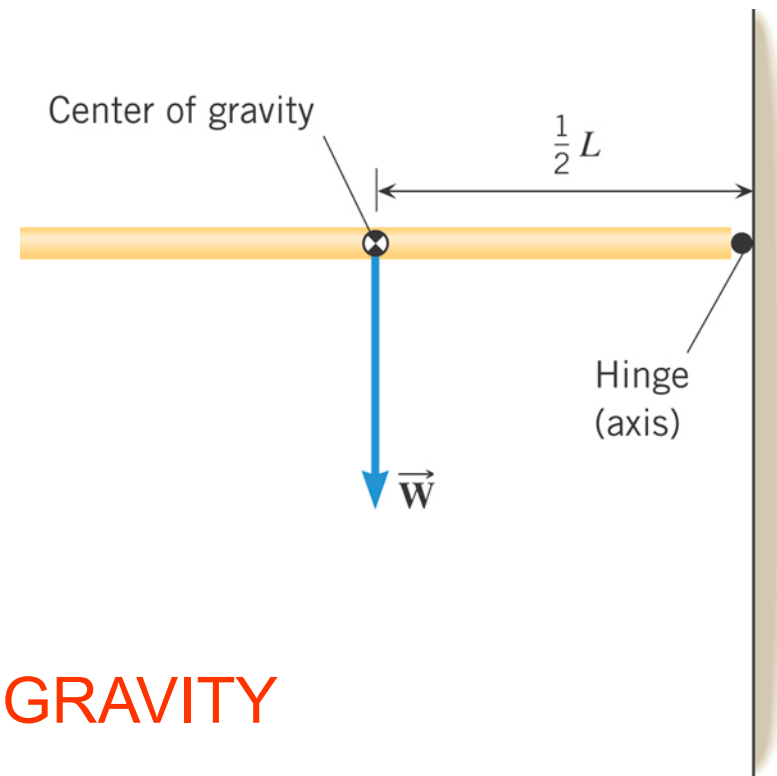
(b) Free-body diagram of the arm



$$\sum \tau = M(\sin 13^\circ)L_M - W_a L_a - W_d L_d = 0$$

$$\begin{aligned} W_d &= \left[+M(\sin 13^\circ)L_M - W_a L_a \right] / L_d \\ &= \left[1840\text{N}(.225)(0.15\text{m}) - 31\text{N}(0.28\text{m}) \right] / 0.62\text{m} \\ &= 86.1\text{N} \end{aligned}$$

9.3 Center of Gravity

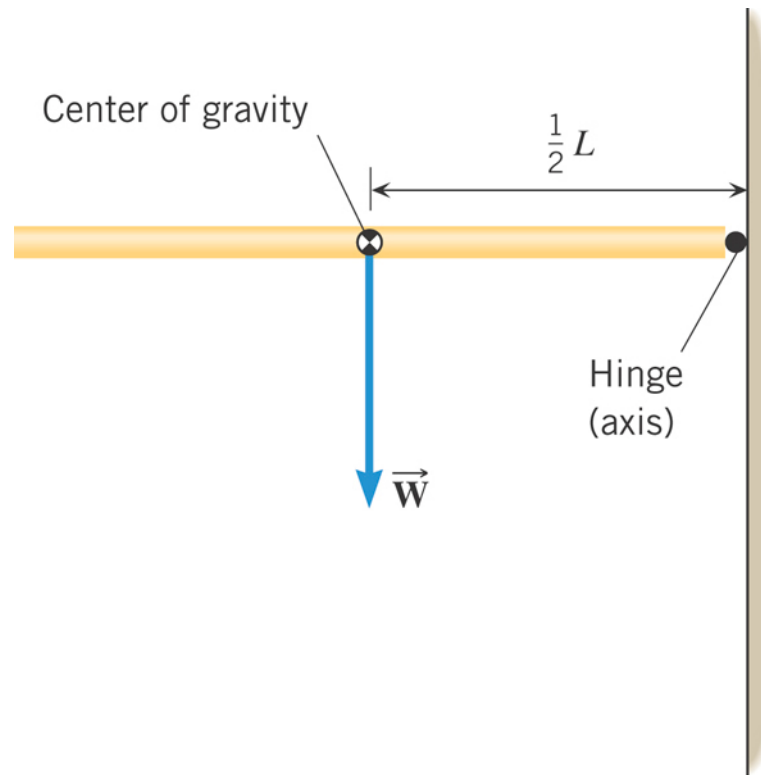


DEFINITION OF CENTER OF GRAVITY

The center of gravity of a rigid body is the point at which its weight can be considered to act when the torque due to the weight is being calculated.

9.3 Center of Gravity

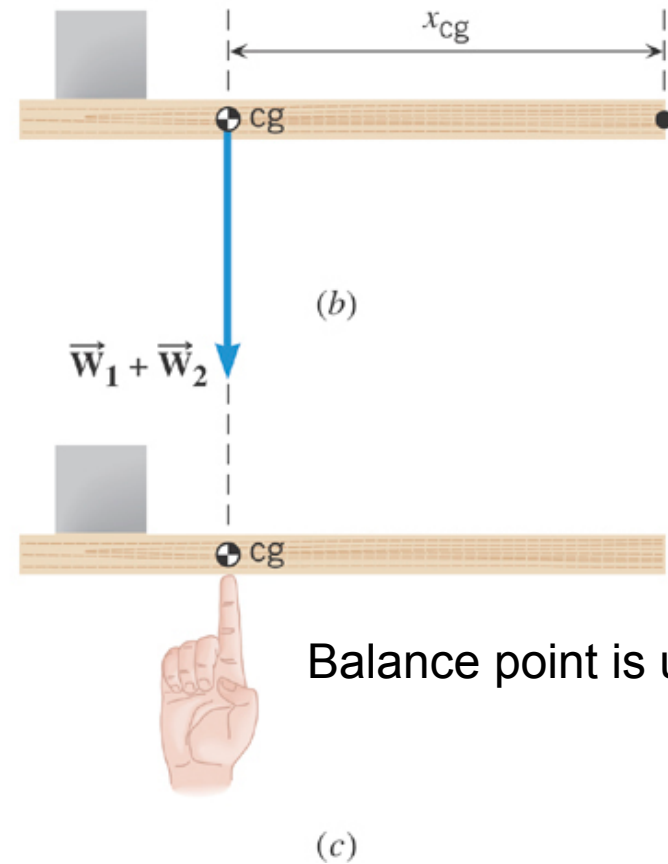
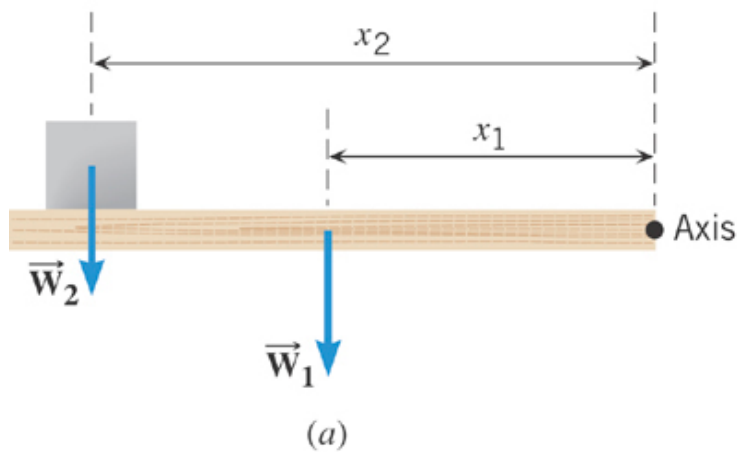
When an object has a symmetrical shape and its weight is distributed uniformly, the center of gravity lies at its geometrical center.



9.3 Center of Gravity

General Form of x_{cg}

$$x_{cg} = \frac{W_1 x_1 + W_2 x_2 + \dots}{W_1 + W_2 + \dots}$$



Center of Gravity, x_{cg} , for 2 masses

$$x_{cg} = \frac{W_1 x_1 + W_2 x_2}{W_1 + W_2}$$

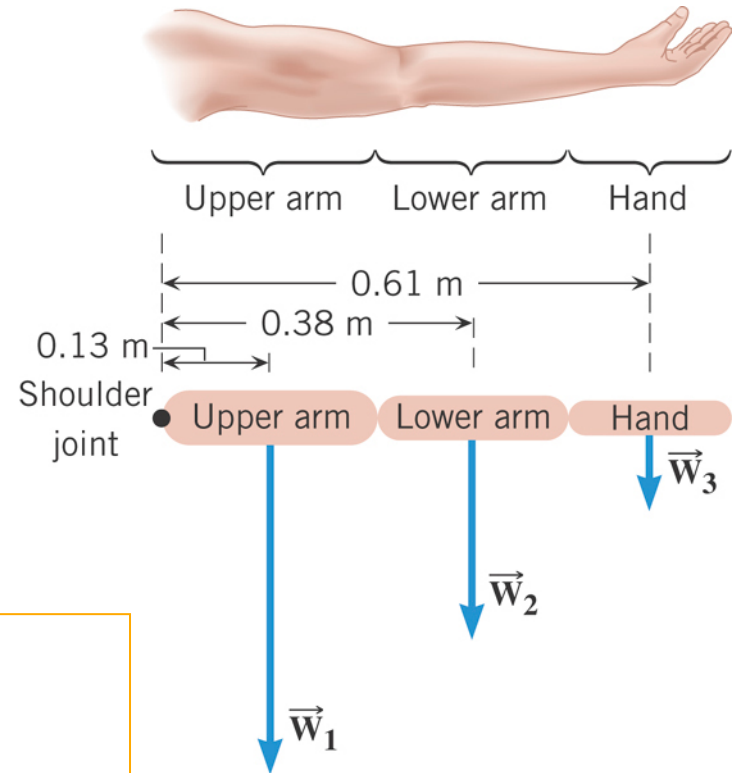
9.3 Center of Gravity

Example 6 The Center of Gravity of an Arm

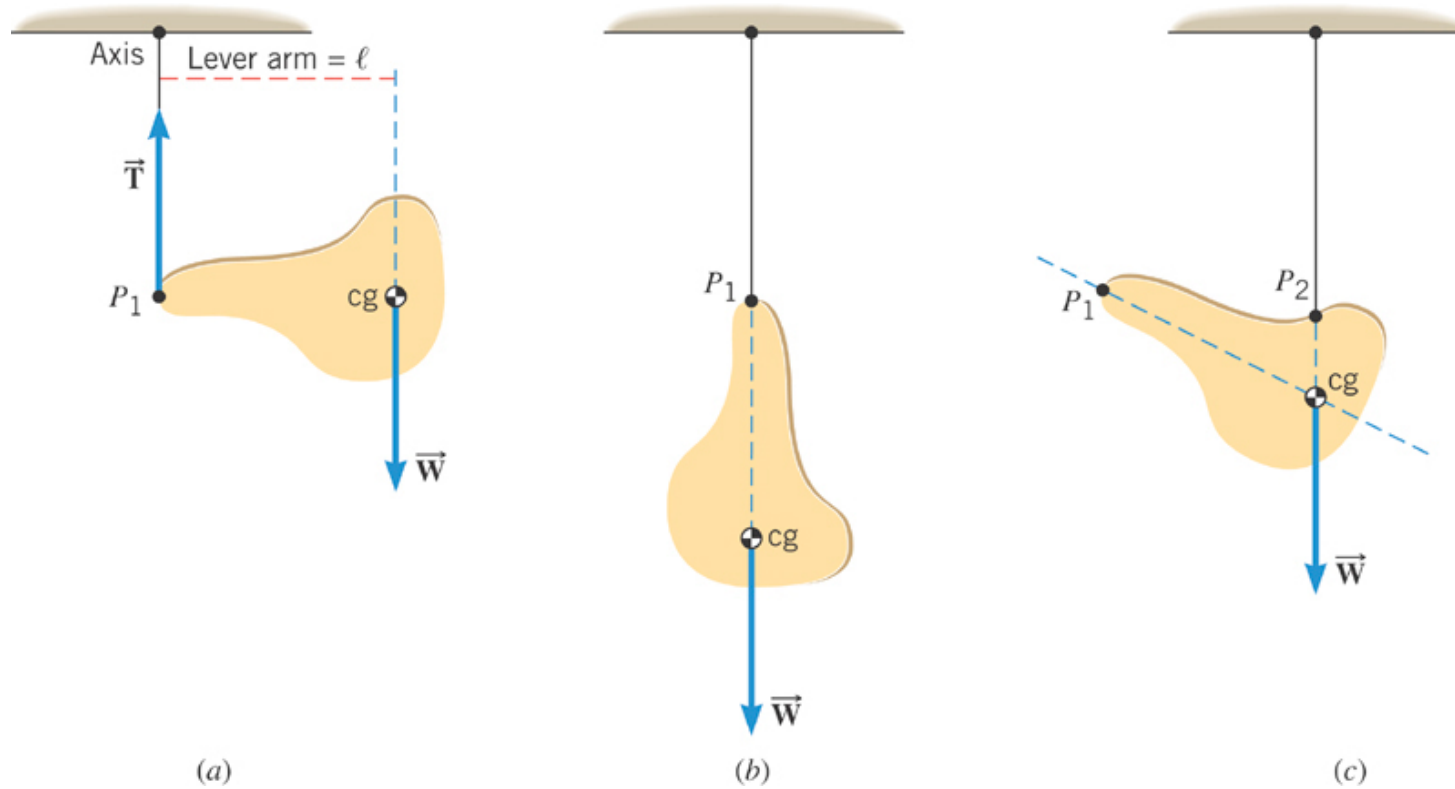
The horizontal arm is composed of three parts: the upper arm (17 N), the lower arm (11 N), and the hand (4.2 N).

Find the center of gravity of the arm relative to the shoulder joint.

$$\begin{aligned}x_{cg} &= \frac{W_1x_1 + W_2x_2 + W_3x_3}{W_1 + W_2 + W_3} \\&= \frac{[17(0.13) + 11(0.38) + 4.2(0.61)] \text{ N} \cdot \text{m}}{(17 + 11 + 4.2) \text{ N}} = 0.28 \text{ m}\end{aligned}$$



9.3 Center of Gravity



Finding the center of gravity of an irregular shape.

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis

$$\tau = F_T r$$

$$= m a_T r$$

$$= m \alpha r^2$$

$$= (m r^2) \alpha$$

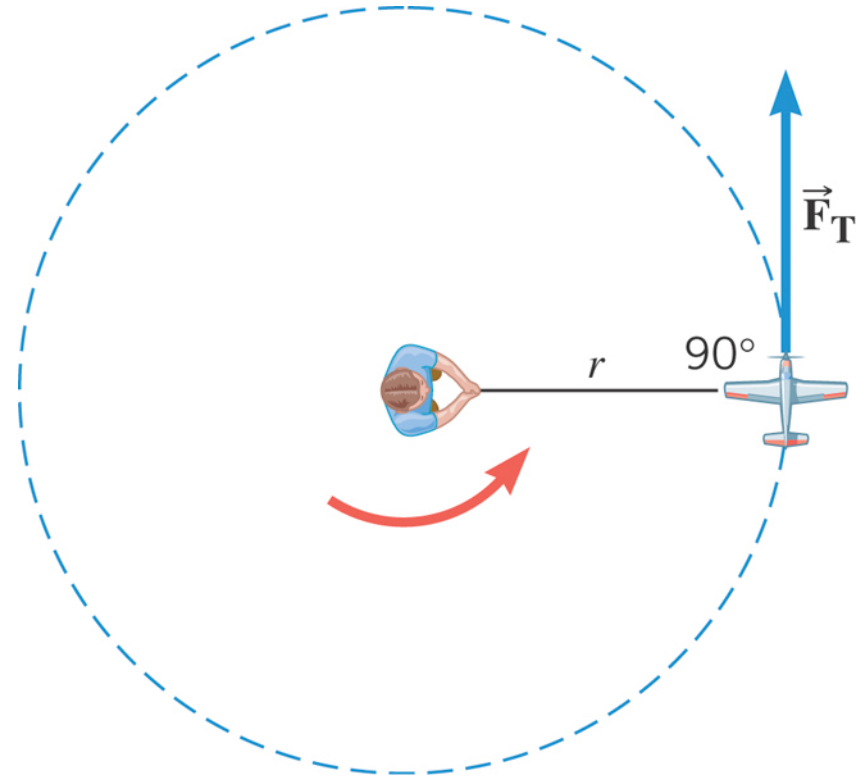
$$= I \alpha$$

$$F_T = m a_T$$

$$a_T = r \alpha$$

$$\text{Let } I = m r^2$$

Moment of Inertia

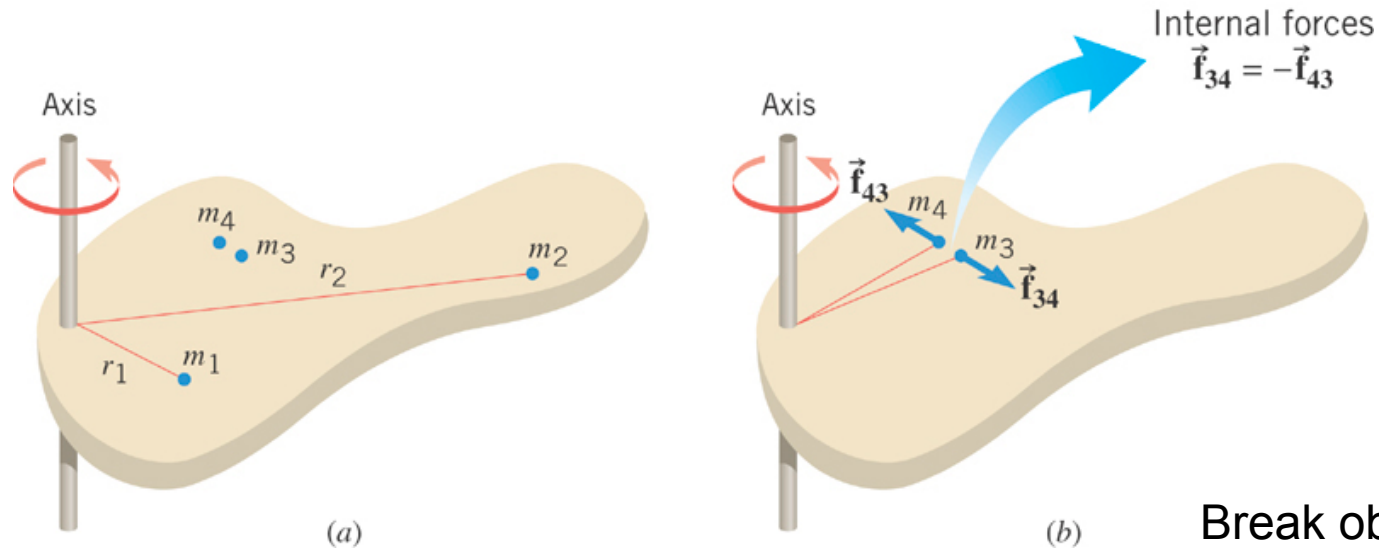


Moment of Inertia, $I = m r^2$, **for a point-mass**, m , at the end of a **massless arm** of length, r .

$$\tau = I \alpha$$

Newton's 2nd Law for rotations

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis



Break object into N individual masses

$$\sum \tau = \sum (mr^2) \alpha$$

Net external
torque

Moment of
inertia

$$\tau_1 = (m_1 r_1^2) \alpha$$

$$\tau_2 = (m_2 r_2^2) \alpha$$

$\vdots$

$$\tau_N = (m_N r_N^2) \alpha$$

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis

ROTATIONAL ANALOG OF NEWTON'S SECOND LAW FOR A RIGID BODY ROTATING ABOUT A FIXED AXIS

$$\text{Net external torque} = \left(\begin{array}{c} \text{Moment of} \\ \text{inertia} \end{array} \right) \times \left(\begin{array}{c} \text{Angular} \\ \text{acceleration} \end{array} \right)$$

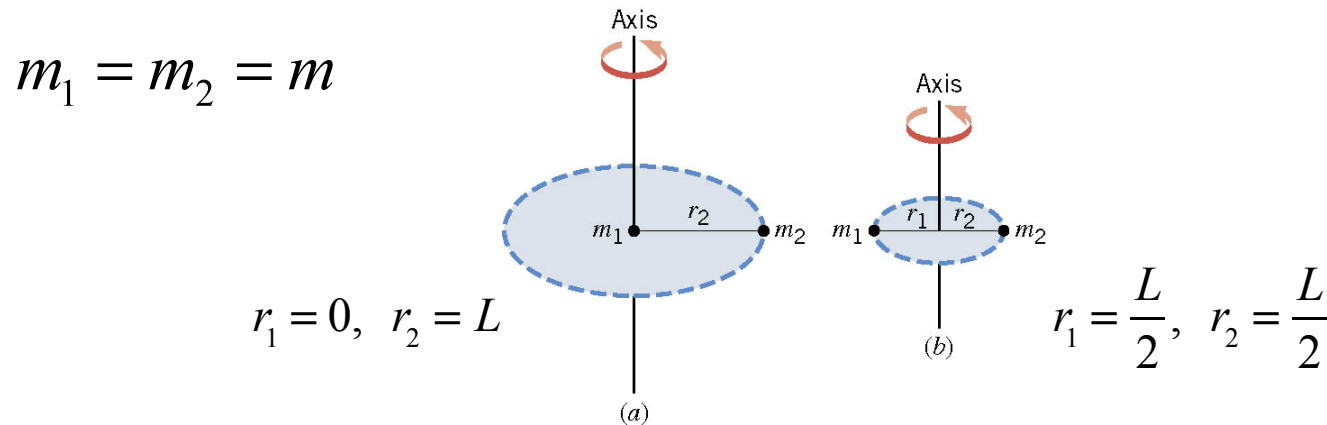
$$\sum \tau = I \alpha \quad \boxed{I = \sum (mr^2)}$$

Requirement: Angular acceleration must be expressed in radians/s².

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis

Moment of Inertia depends on axis of rotation.

Two particles each with mass, m , and are fixed at the ends of a thin rigid rod. The length of the rod is L . Find the moment of inertia when this object rotates relative to an axis that is perpendicular to the rod at (a) one end and (b) the center.



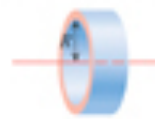
$$\begin{aligned} \text{(a)} \quad I &= \sum (mr^2) = m_1 r_1^2 + m_2 r_2^2 = m(0)^2 + m(L)^2 \\ &= mL^2 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad I &= \sum (mr^2) = m_1 r_1^2 + m_2 r_2^2 = m\left(\frac{L}{2}\right)^2 + m\left(\frac{L}{2}\right)^2 \\ &= \frac{1}{2}mL^2 \end{aligned}$$

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis

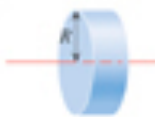
Table 9.1 Moments of Inertia I for Various Rigid Objects of Mass M

Thin-walled hollow cylinder or hoop



$$I = MR^2$$

Solid cylinder or disk



$$I = \frac{1}{2}MR^2$$

Thin rod, axis perpendicular to rod and passing through center



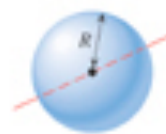
$$I = \frac{1}{12}ML^2$$

Thin rod, axis perpendicular to rod and passing through one end



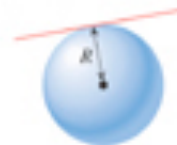
$$I = \frac{1}{3}ML^2$$

Solid sphere, axis through center



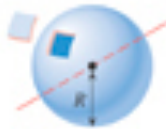
$$I = \frac{2}{5}MR^2$$

Solid sphere, axis tangent to surface



$$I = \frac{7}{5}MR^2$$

Thin-walled spherical shell, axis through center



$$I = \frac{2}{3}MR^2$$

Thin rectangular sheet, axis parallel to one edge and passing through center of other edge



$$I = \frac{1}{12}ML^2$$

Thin rectangular sheet, axis along one edge



$$I = \frac{1}{3}ML^2$$

$$I = MR^2$$

$$I = \frac{1}{2}MR^2$$

$$I = \frac{1}{12}MR^2$$

$$I = \frac{1}{3}MR^2$$

$$I = \frac{2}{5}MR^2$$

$$I = \frac{7}{5}MR^2$$

$$I = \frac{2}{3}MR^2$$

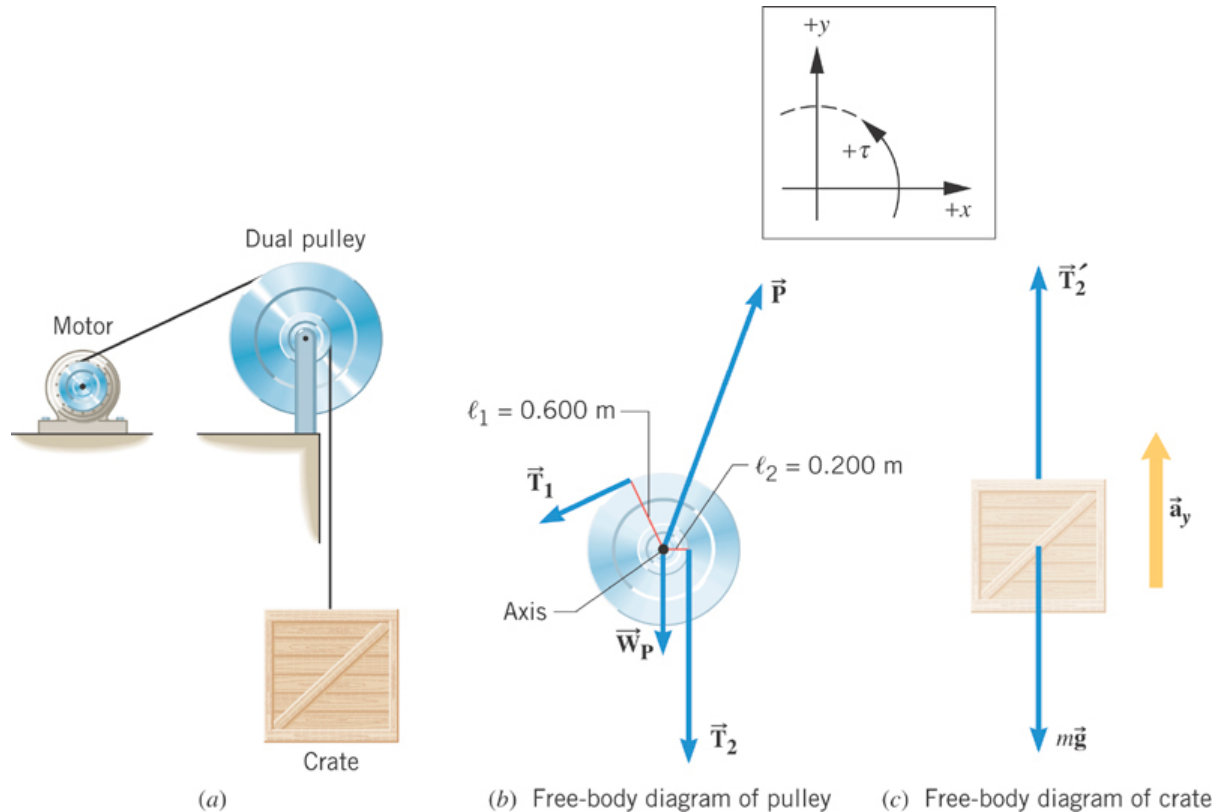
$$I = \frac{1}{12}MR^2$$

$$I = \frac{1}{3}MR^2$$

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis

Example 12 Hoisting a Crate

The combined moment of inertia of the dual pulley is $50.0 \text{ kg}\cdot\text{m}^2$. The crate weighs 4420 N . A tension of 2150 N is maintained in the cable attached to the motor. Find the angular acceleration of the dual pulley.



9.4 Newton's Second Law for Rotational Motion About a Fixed Axis

2nd law for linear motion of crate

$$\sum F_y = T_2 - mg = ma_y \quad a_y = \ell_2 \alpha$$

2nd law for rotation of the pulley

$$\sum \tau = T_1 \ell_1 - T_2 \ell_2 = I \alpha$$

$$I = 46 \text{ kg} \cdot \text{m}^2$$

$$mg = 4420 \text{ N}$$

$$T_1 = 2150 \text{ N}$$

$$T_2 - mg = m \ell_2 \alpha$$

$$T_1 \ell_1 - T_2 \ell_2 = I \alpha$$

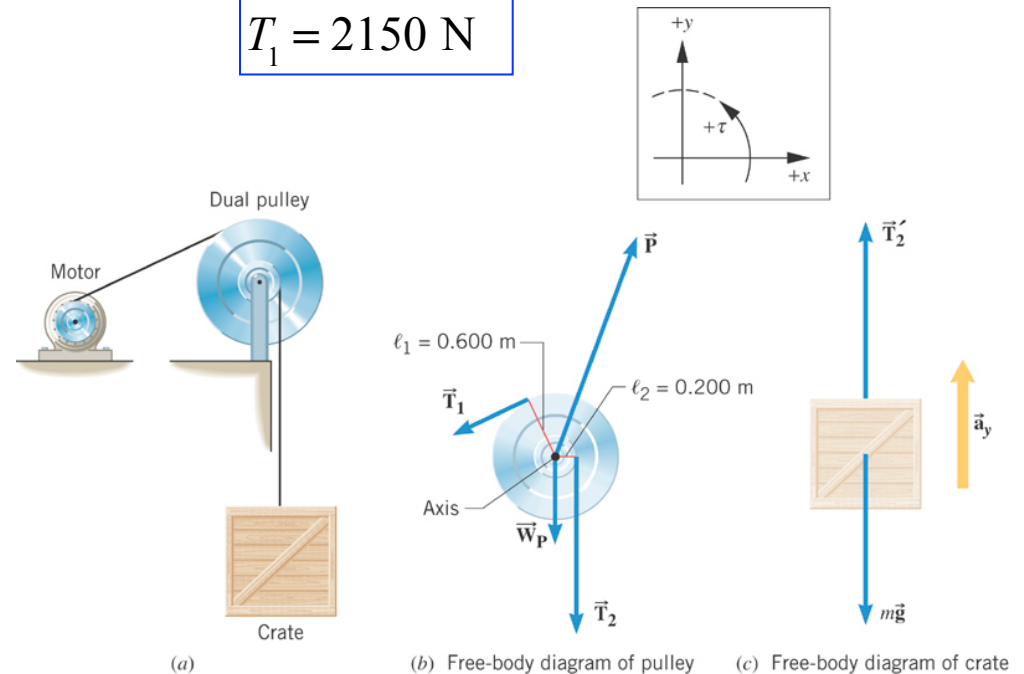
$$T_2 = m \ell_2 \alpha + mg$$

$$T_2 = \frac{T_1 \ell_1 - I \alpha}{\ell_2}$$

$$m \ell_2^2 \alpha + I \alpha = T_1 \ell_1 - mg \ell_2$$

$$\alpha = \frac{T_1 \ell_1 - mg \ell_2}{m \ell_2^2 + I} = 6.3 \text{ rad/s}^2$$

$$T_2 = m \ell_2 \alpha + mg = (451(.6)(6.3) + 4420) \text{ N} = 6125 \text{ N}$$



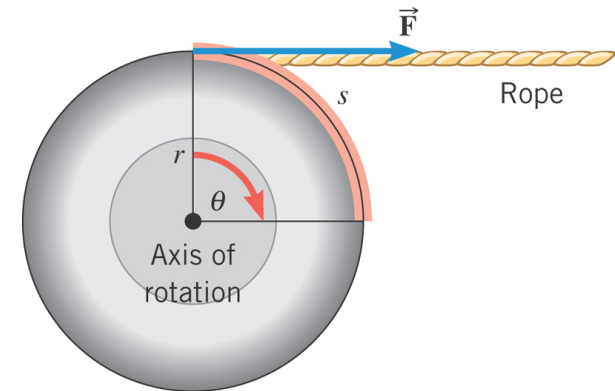
$$\vec{T}_2' = +T_2$$

$$\vec{T}_2 = -T_2$$

9.5 Rotational Work and Energy

Work
rotating
a mass:

$$\begin{aligned} W &= Fs & s &= r\theta \\ &= Fr\theta & Fr &= \tau \\ &= \tau\theta \end{aligned}$$



DEFINITION OF ROTATIONAL WORK

The rotational work done by a constant torque in turning an object through an angle is

$$W_R = \tau\theta$$

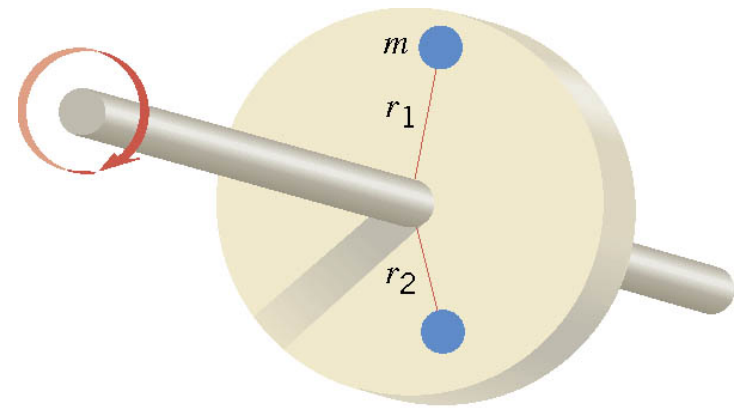
Requirement: The angle must be expressed in radians.

SI Unit of Rotational Work: joule (J)

9.5 Rotational Work and Energy

Kinetic Energy of a rotating **one** point mass

$$\begin{aligned} KE &= \frac{1}{2} m v_T^2 & v_T &= r\omega \\ &= \frac{1}{2} m r^2 \omega^2 \end{aligned}$$



Kinetic Energy of many rotating point masses

$$KE = \sum \left(\frac{1}{2} m r^2 \omega^2 \right) = \frac{1}{2} \left(\sum m r^2 \right) \omega^2 = \frac{1}{2} I \omega^2$$

DEFINITION OF ROTATIONAL KINETIC ENERGY

The rotational kinetic energy of a rigid rotating object is

$$KE_R = \frac{1}{2} I \omega^2$$

Requirement: The angular speed must be expressed in rad/s.

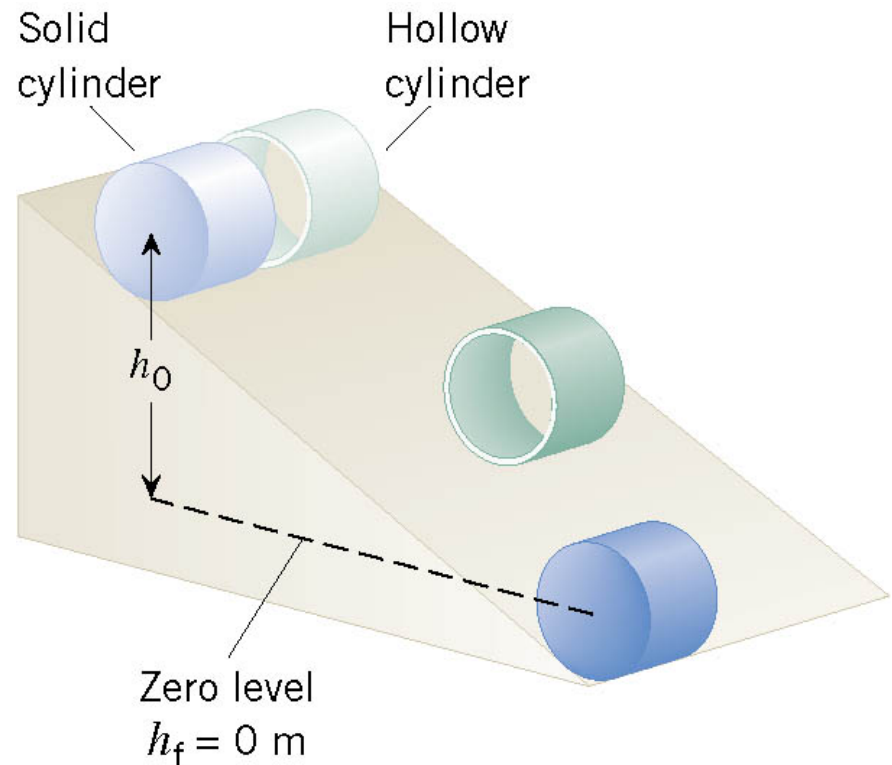
SI Unit of Rotational Kinetic Energy: joule (J)

9.5 Rotational Work and Energy

Example 13 Rolling Cylinders

A thin-walled hollow cylinder (mass = m_h , radius = r_h) and a solid cylinder (mass = m_s , radius = r_s) start from rest at the top of an incline.

Determine which cylinder has the greatest translational speed upon reaching the bottom.



9.5 Rotational Work and Energy

Total Energy = Kinetic Energy + Rotational Energy + Potential Energy

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgh$$

ENERGY CONSERVATION $E_f = E_i$

$$\frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 + mgh_f = \frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 + mgh_i$$

$$\omega_f = v_f / r$$

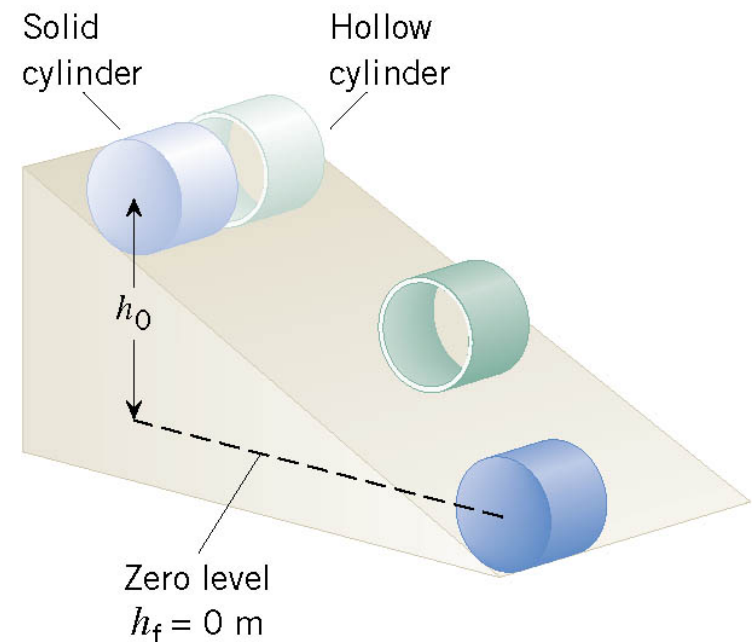
$$0 \quad 0 \quad 0$$

$$\frac{1}{2}mv_f^2 + \frac{1}{2}Iv_f^2 / r^2 = mgh_i$$

$$v_f = \sqrt{\frac{2mgh_o}{m + I/r^2}}$$

The cylinder with the **smaller** moment of inertia will have a **greater** final translational speed.

Same mass for cylinder and hoop



9.6 *Angular Momentum*

DEFINITION OF ANGULAR MOMENTUM

The angular momentum L of a body rotating about a fixed axis is the product of the body's moment of inertia and its angular velocity with respect to that axis:

$$L = I\omega$$

Requirement: The angular speed must be expressed in rad/s.

SI Unit of Angular Momentum: $\text{kg}\cdot\text{m}^2/\text{s}$

9.6 Angular Momentum

PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM

The angular momentum of a system remains constant (is conserved) if the net external torque acting on the system is zero.

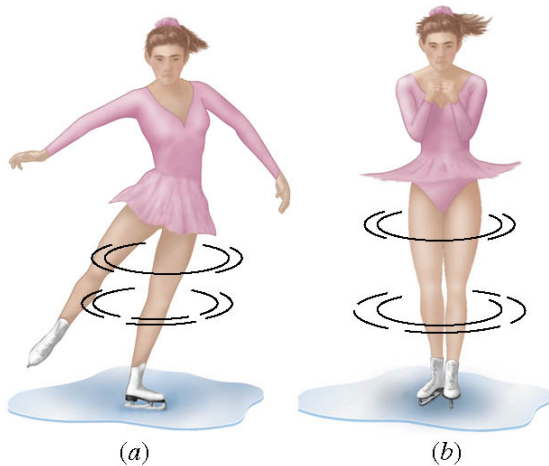
Angular momentum, L

$$L_i = I_i \omega_i; \quad L_f = I_f \omega_f$$

No external torque

$\Rightarrow$ Angular momentum conserved

$$L_f = L_i$$



Moment of Inertia
decreases

$$I = \sum mr^2, \quad r_f < r_i$$

$$I_f < I_i$$

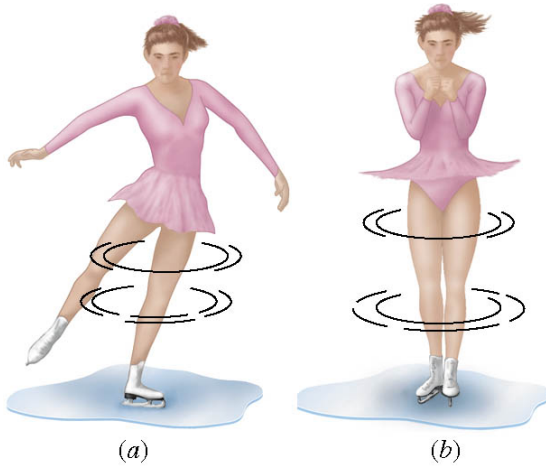
$$\frac{I_i}{I_f} > 1$$

$$I_f \omega_f = I_i \omega_i$$

$$\omega_f = \frac{I_i}{I_f} \omega_i; \quad \frac{I_i}{I_f} > 1$$

$$\omega_f > \omega_i \text{ (angular speed increases)}$$

9.6 Angular Momentum



From Angular Momentum Conservation

$$\omega_f = \left(I_i / I_f \right) \omega_i \quad I_i / I_f > 1$$

Angular velocity increases

Is Energy conserved?

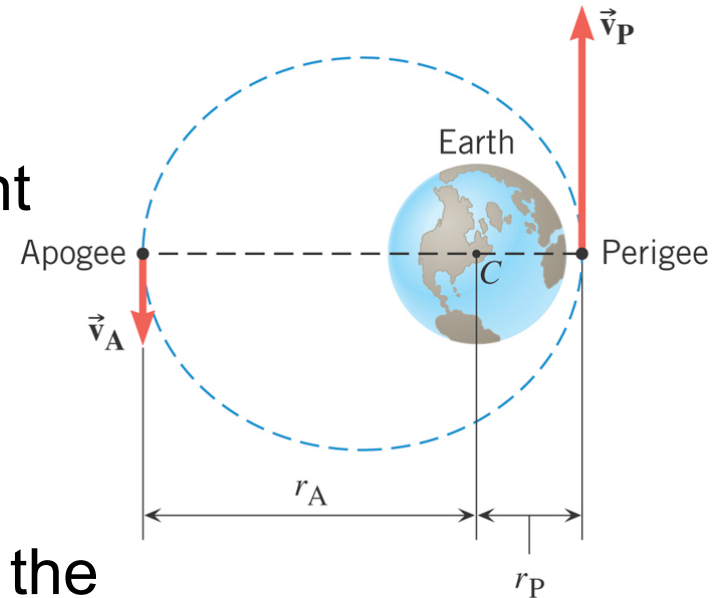
$$\begin{aligned} KE_f &= \frac{1}{2} I_f \omega_f^2 \\ &= \frac{1}{2} I_f \left(I_i / I_f \right)^2 \omega_i^2 \\ &= \left(I_i / I_f \right) \left(\frac{1}{2} I_i \omega_i^2 \right) \quad KE_i = \frac{1}{2} I_i \omega_i^2; \\ &= \left(I_i / I_f \right) KE_i \Rightarrow \text{Kinetic Energy increases} \end{aligned}$$

Energy is **not conserved** because pulling in the arms does (NC) work on their mass and increases the kinetic energy of rotation

9.6 Angular Momentum

Example 15 A Satellite in an Elliptical Orbit

An artificial satellite is placed in an elliptical orbit about the earth. Its point of closest approach is $8.37 \times 10^6 \text{ m}$ from the center of the earth, and its point of greatest distance is $25.1 \times 10^6 \text{ m}$ from the center of the earth. The speed of the satellite at the perigee is 8450 m/s . Find the speed at the apogee.



Gravitational force along L (no torque)

Angular momentum conserved

$$I_A \omega_A = I_P \omega_P$$

$$mr_A^2 (v_A / r_A) = mr_P^2 (v_P / r_P)$$

$$r_A v_A = r_P v_P \Rightarrow v_A = (r_P / r_A) v_P = [(8.37 \times 10^6) / (25.1 \times 10^6)] (8450 \text{ m/s}) = 2820 \text{ m/s}$$

$$I_A = mr_A^2; \quad I_P = mr_P^2$$

$$\omega_A = v_A / r_A; \quad \omega_P = v_P / r_P$$