Chapter 9

Rotational Dynamics

9.1 *The Action of Forces and Torques on Rigid Objects*

Chapter 8 developed the concepts of angular motion.

θ : angles and radian measure for angular variables

ω : angular velocity of rotation (same for entire object)

α : angular acceleration (same for entire object)

$v_T = \omega r$: tangential velocity

$a_T = \alpha r$: tangential acceleration

According to Newton's second law, a net force causes an object to have a ***linear acceleration***.

What causes an object to have an ***angular acceleration***?

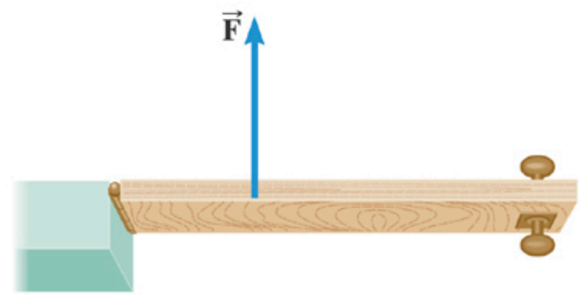
TORQUE

9.1 The Action of Forces and Torques on Rigid Objects

The amount of torque depends on where and in what direction the force is applied, as well as the location of the axis of rotation.

(a)

Maximum rotational effect of the force F.



(b)

Smaller rotational effect of the force F.

(c)

Rotational effect of the force F is minimal; it compresses more than rotates the bar

9.1 The Action of Forces and Torques on Rigid Objects

DEFINITION OF TORQUE

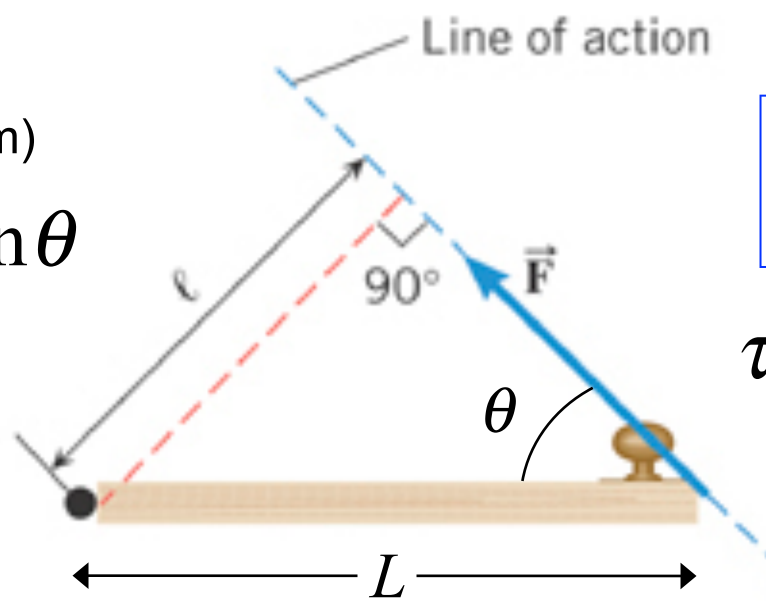
Magnitude of Torque = (Magnitude of the force) x (Lever arm)

$$\tau = F\ell$$

Direction: The torque is positive when the force tends to produce a counterclockwise rotation about the axis.
SI Unit of Torque: newton x meter (N·m)

(Lever arm)

$$\ell = L\sin\theta$$



θ is the angle between $\vec{\mathbf{F}}$ and L

$$\tau = F\ell = FL\sin\theta$$

9.1 The Action of Forces and Torques on Rigid Objects

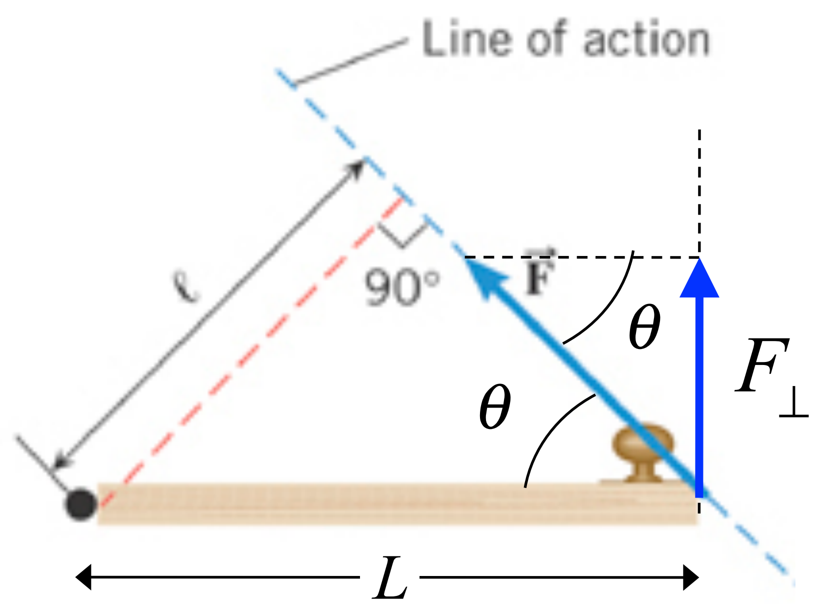
$$\text{Magnitude of Torque} = \left(\text{Magnitude of the force}\right) \times \left(\text{Lever arm}\right)$$

$$\tau = F\ell = FL\sin\theta \qquad \ell = L\sin\theta$$

Alternate (Equivalent) Interpretation

$$\text{Magnitude of Torque} = \left(\text{Component of Force} \perp \text{to } L\right) \times L$$

$$\tau = F_{\perp}L = \left(F\sin\theta\right)(L) = FL\sin\theta$$

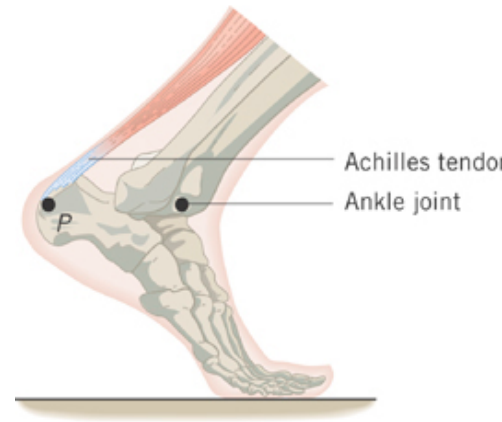


$$F_{\perp} = F\sin\theta$$

θ is the angle between $\vec{\mathbf{F}}$ and L

Example 2 The Achilles Tendon

The tendon exerts a force of magnitude 720 N. Determine the torque (magnitude and direction) of this force about the ankle joint.



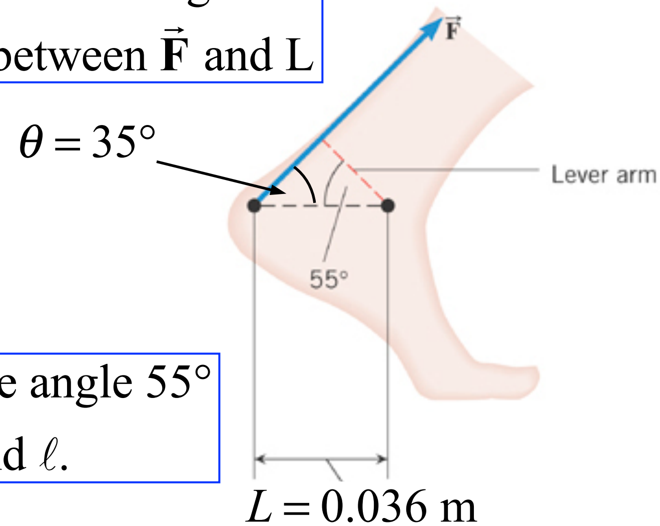
(a)

$$\tau = \left(F\sin\theta\right)L = (720\text{ N})(\sin 35°)(.036\text{ m})$$
$$= 15.0\text{ N}\cdot\text{m}$$

θ is the angle between $\vec{\mathbf{F}}$ and L

$$\tau = F\left(L\sin\theta\right) = (720\text{ N})(.036\text{ m})\sin 35°$$
$$= 15.0\text{ N}\cdot\text{m}$$

Example gave angle 55° between L and ℓ.



(b)

Direction is clockwise (–) around ankle joint

Torque vector $\tau = -15.0\text{ N}\cdot\text{m}$

9.2 Rigid Objects in Equilibrium

If a rigid body is in equilibrium, neither its linear motion nor its rotational motion changes.

$$a_x = a_y = 0 \qquad \alpha = 0$$

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum \tau = 0$$

All equilibrium problems use these equations – no net force and no net torque.

EQUILIBRIUM OF A RIGID BODY

A rigid body is in equilibrium if it has zero translational acceleration and zero angular acceleration. In equilibrium, the sum of the externally applied forces is zero, and the sum of the externally applied torques is zero.

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum \tau = 0$$

Note: constant linear speed or constant rotational speed are allowed for an object in equilibrium.

Reasoning Strategy

1. Select the object to which the equations for equilibrium are to be applied.

2. Draw a free-body diagram that shows all of the external forces acting on the object.

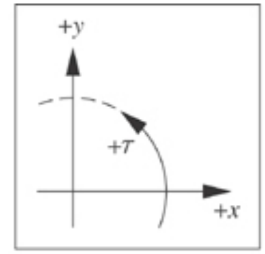
3. Choose a convenient set of *x*, *y* axes and resolve all forces into components that lie along these axes.

4. Apply the equations that specify the balance of forces at equilibrium. (Set the net force in the *x* and *y* directions equal to zero.)

5. Select a convenient axis of rotation. Set the sum of the torques about this axis equal to zero.

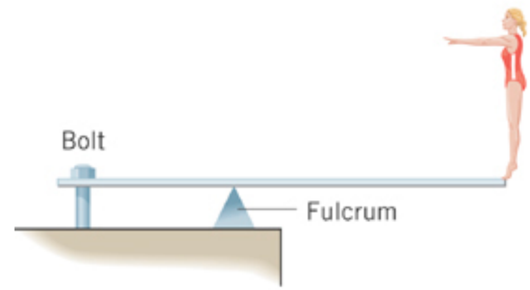
6. Solve the equations for the desired unknown quantities.

Example 3 A Diving Board



Counter-clockwise torque is positive

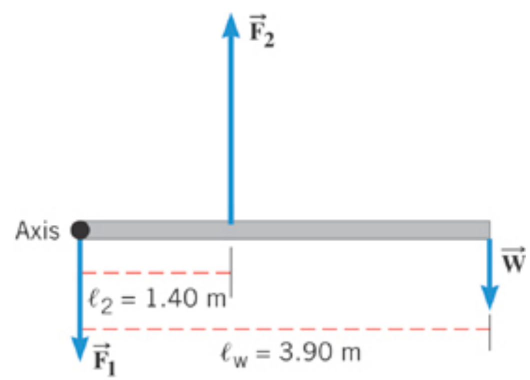
A woman whose weight is 530 N is poised at the right end of a diving board with length 3.90 m. The board has negligible weight and is supported by a fulcrum 1.40 m away from the left end. Find the forces that the bolt and the fulcrum exert on the board.



(*a*)

F_1 acts on rotation axis - produces no torque.

$$\sum \tau = 0 = F_2 \ell_2 - W \ell_W$$
$$F_2 = W(\ell_W / \ell_2) = 530\text{N}(3.9/1.4) = 1480 \text{ N}$$



(*b*) Free-body diagram of the diving board

$$\sum F_y = 0 = -F_1 + F_2 - W$$
$$F_1 = F_2 - W = (1480 - 530)\text{N} = 950 \text{ N}$$

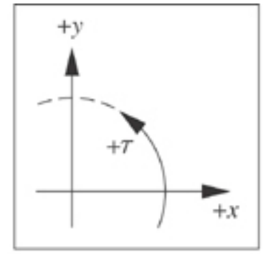
9.2 Rigid Objects in Equilibrium

Choice of pivot is arbitary (most convenient)

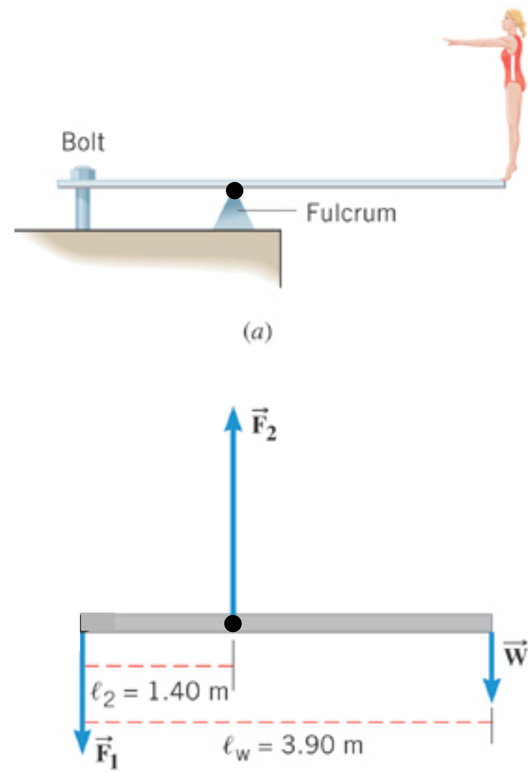
Pivot at fulcum: F_2 produces no torque.

$$\sum \tau = 0 = F_1 \ell_2 - W(\ell_W - \ell_2)$$
$$F_1 = W(\ell_W / \ell_2 - 1) = (530\text{N})(1.8) = 950\,\text{N}$$

$$\sum F_y = 0 = -F_1 + F_2 - W$$
$$F_2 = F_1 + W = \left(950 + 530\right)\text{N} = 1480\ \text{N}$$



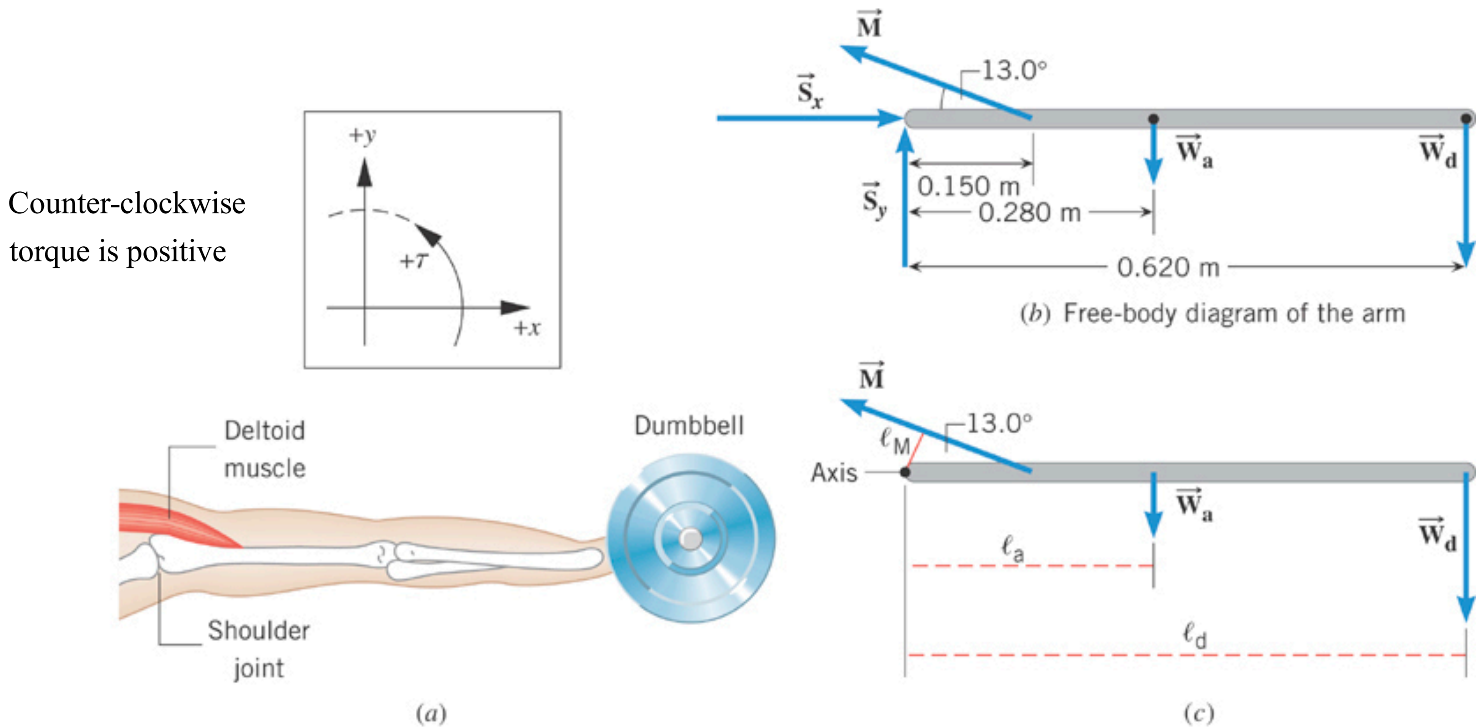
Counter-clockwise torque is positive



(a)

(b) Free-body diagram of the diving board

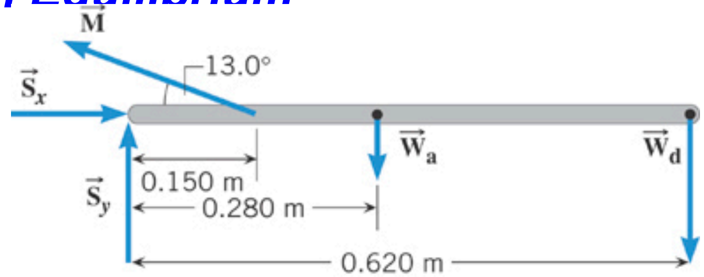
9.2 Rigid Objects in Equilibrium



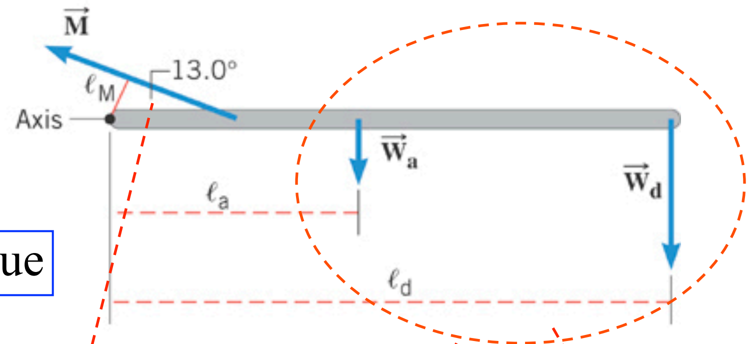
Example 5 Bodybuilding

The arm is horizontal and weighs 31.0 N. The deltoid muscle can supply 1840 N of force. What is the weight of the heaviest dumbell he can hold?

9.2 Rigid Objects in Equilibrium



(*b*) Free-body diagram of the arm



(*c*)

positive torque

negative torques

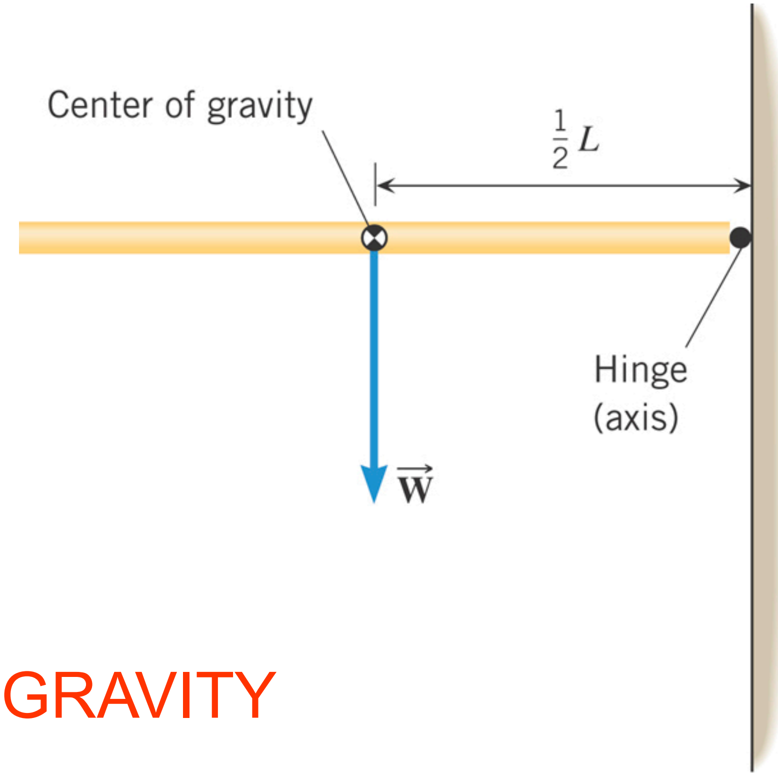
$$\sum \tau = M\left(\sin 13°\right) L_M - W_a L_a - W_d L_d = 0$$

$$W_d = \left[+M\left(\sin 13°\right) L_M - W_a L_a\right] / L_d$$

$$= \left[1840\text{N}(.225)(0.15\text{m}) - 31\text{N}(0.28\text{m})\right] / 0.62\text{m}$$

$$= 86.1\text{N}$$

9.3 Center of Gravity

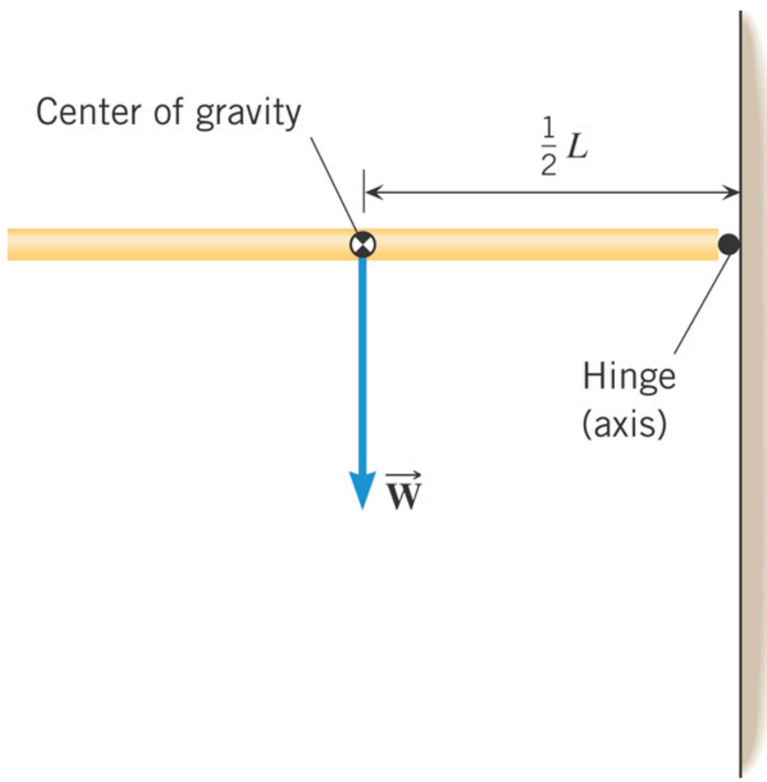


DEFINITION OF CENTER OF GRAVITY

The center of gravity of a rigid body is the point at which its weight can be considered to act when the torque due to the weight is being calculated.

9.3 Center of Gravity

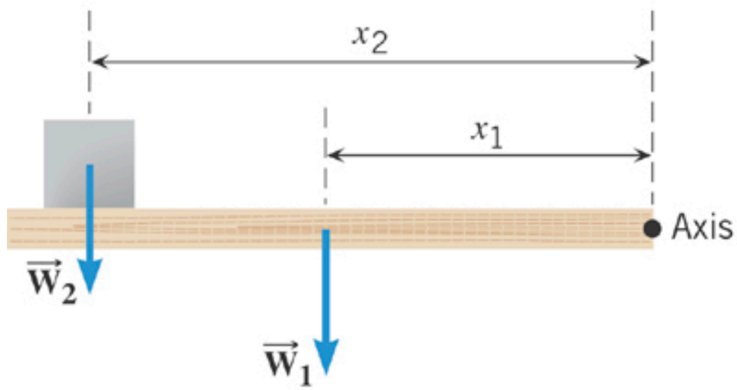
When an object has a symmetrical shape and its weight is distributed uniformly, the center of gravity lies at its geometrical center.



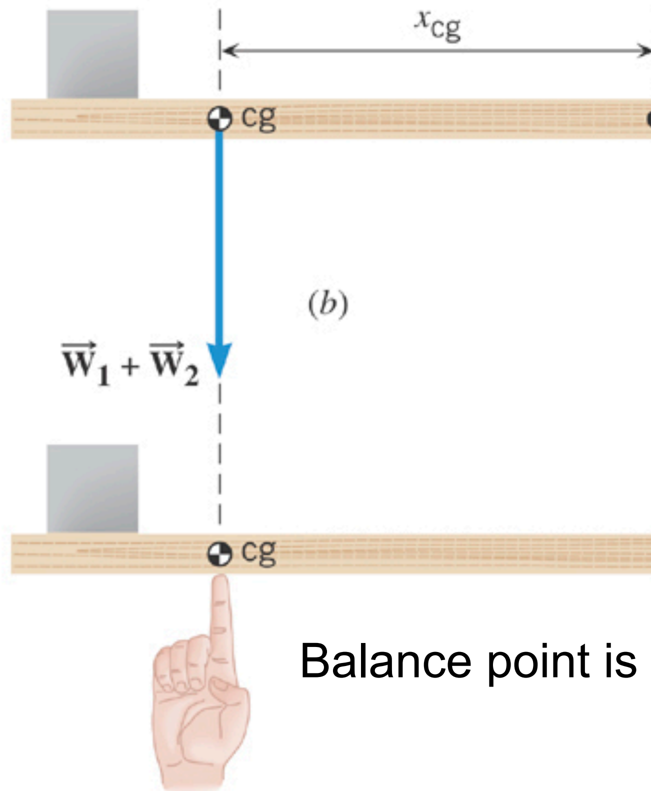
9.3 Center of Gravity

General Form of x_{cg}

$$x_{cg} = \frac{W_1 x_1 + W_2 x_2 + \cdots}{W_1 + W_2 + \cdots}$$



(a)



(b)

Balance point is under x_{cg}

(c)

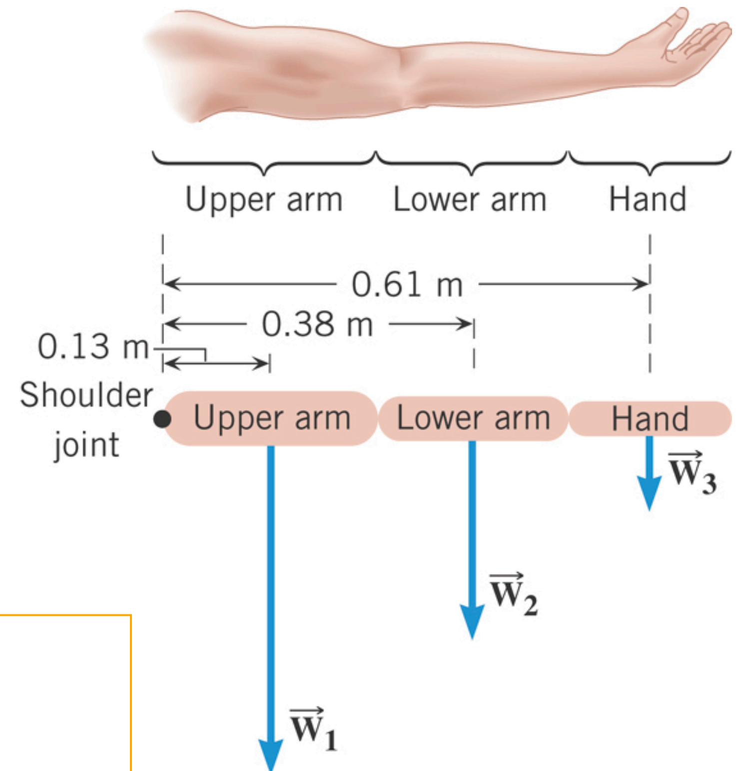
Center of Gravity, x_{cg} , for 2 masses

$$x_{cg} = \frac{W_1 x_1 + W_2 x_2}{W_1 + W_2}$$

Example 6 The Center of Gravity of an Arm

The horizontal arm is composed of three parts: the upper arm (17 N), the lower arm (11 N), and the hand (4.2 N).

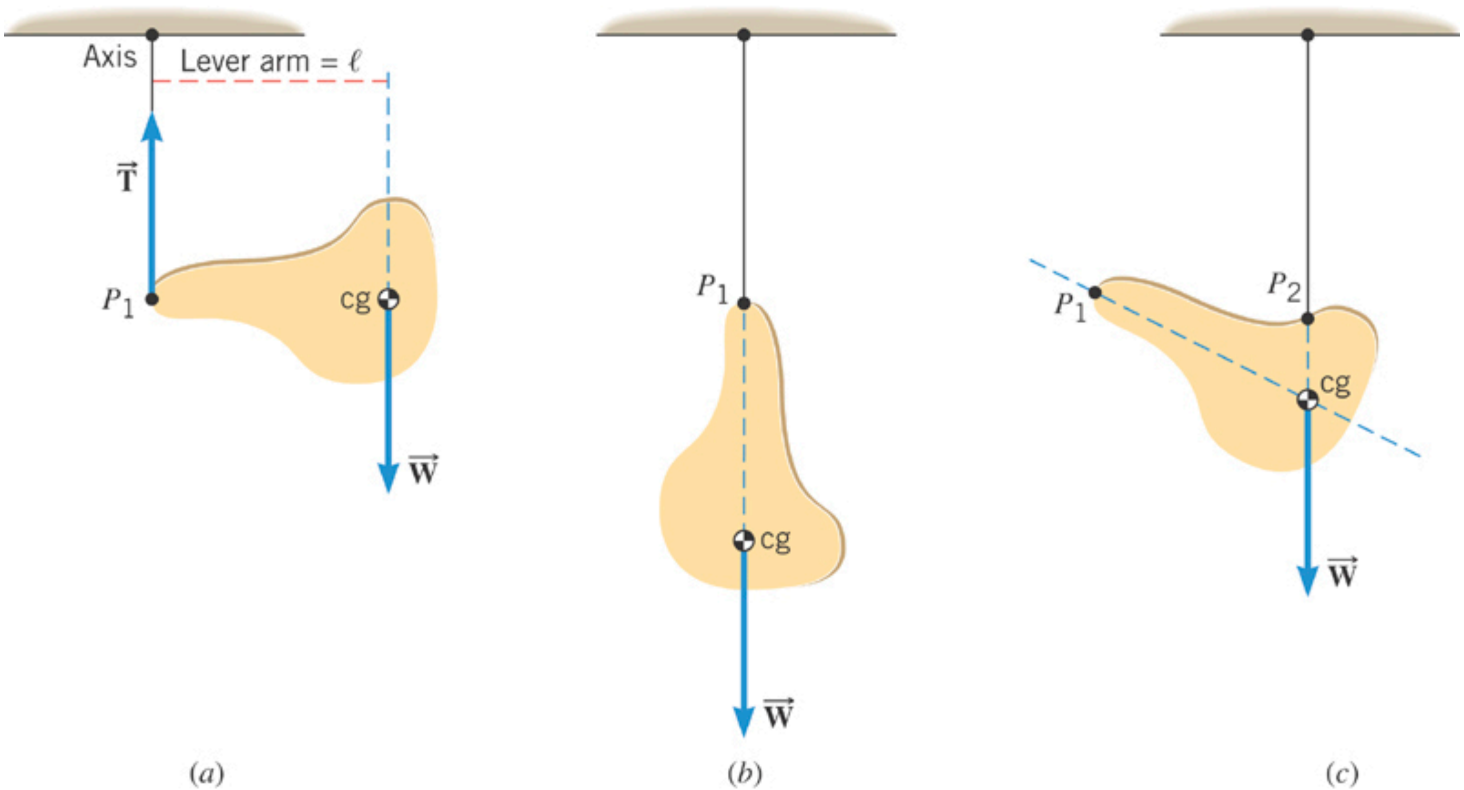
Find the center of gravity of the arm relative to the shoulder joint.



$$x_{cg} = \frac{W_1 x_1 + W_2 x_2 + W_3 x_3}{W_1 + W_2 + W_3}$$

$$= \frac{\left[17(0.13) + 11(0.38) + 4.2(0.61)\right]\text{N}\cdot\text{m}}{(17 + 11 + 4.2)\ \text{N}} = 0.28\ \text{m}$$

9.3 Center of Gravity



Finding the center of gravity of an irregular shape.

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis

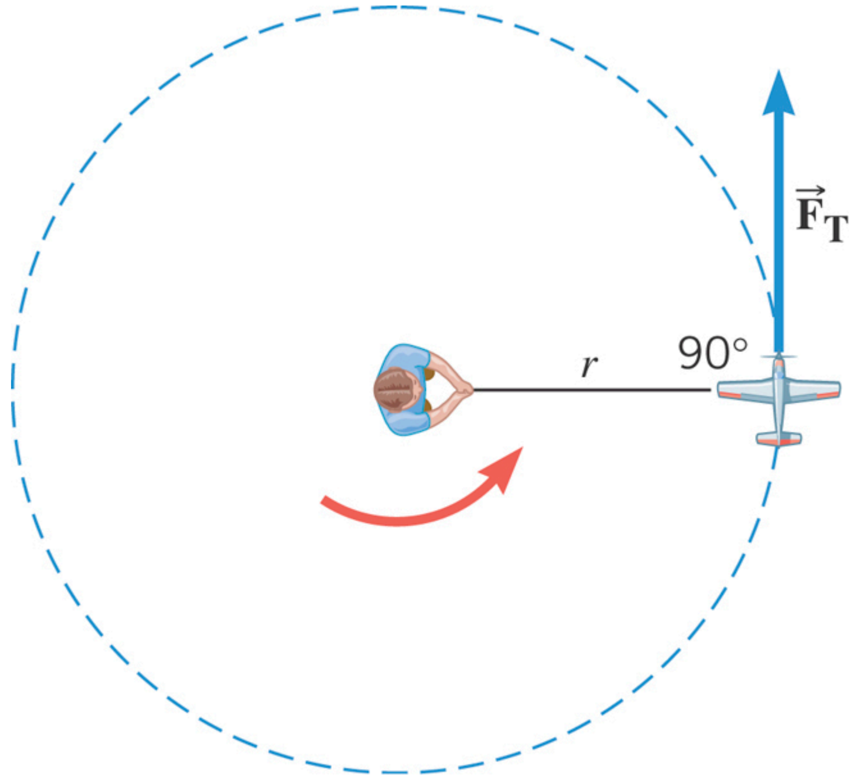
$$\tau = F_T r$$
$$= m a_T r$$
$$= m\alpha r^2$$
$$= \left(mr^2\right)\alpha$$
$$= I\alpha$$

$$\boxed{F_T = m a_T}$$

$$\boxed{a_T = r\alpha}$$

$$\boxed{\text{Let } I = mr^2}$$

Moment of Inertia

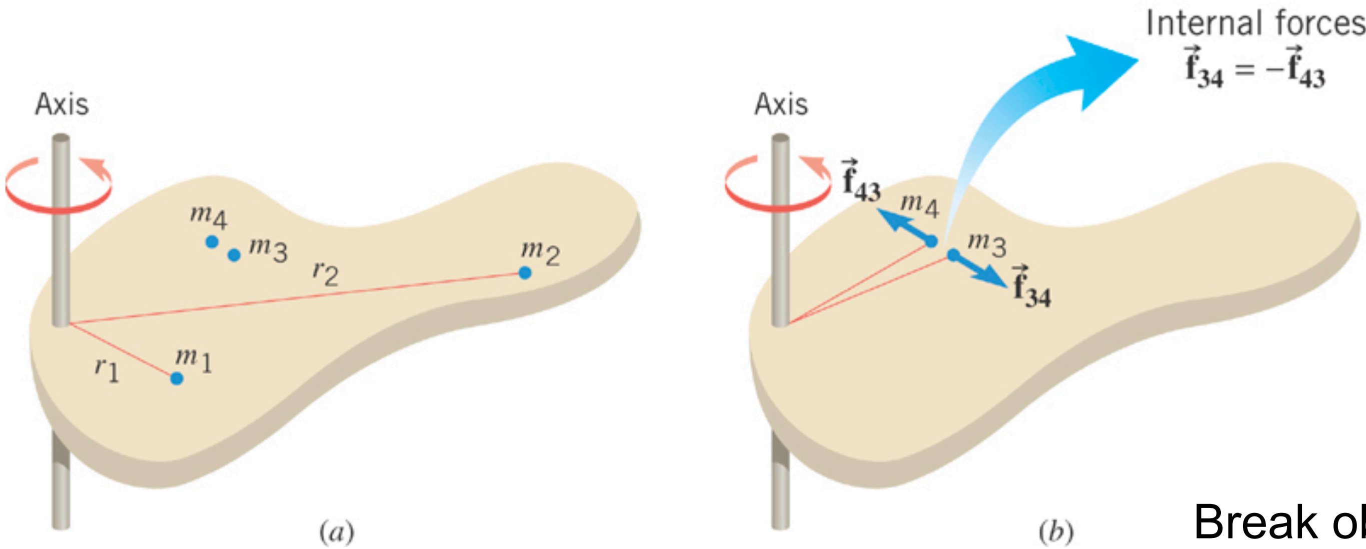


Moment of Inertia, $I = mr^2$, for a point-mass, m, at the end of a massless arm of length, r.

$$\boxed{\tau = I\alpha}$$

Newton's 2nd Law for rotations

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis



Break object into N individual masses

$$\sum \tau = \sum \left(mr^2 \right) \alpha$$

Net external torque

Moment of inertia

$$\tau_1 = \left(m_1 r_1^2 \right) \alpha$$

$$\tau_2 = \left(m_2 r_2^2 \right) \alpha$$

$$\vdots$$

$$\tau_N = \left(m_N r_N^2 \right) \alpha$$

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis

ROTATIONAL ANALOG OF NEWTON'S SECOND LAW FOR A RIGID BODY ROTATING ABOUT A FIXED AXIS

$$\text{Net external torque} = \begin{pmatrix}\text{Moment of} \\ \text{inertia}\end{pmatrix} \times \begin{pmatrix}\text{Angular} \\ \text{acceleration}\end{pmatrix}$$

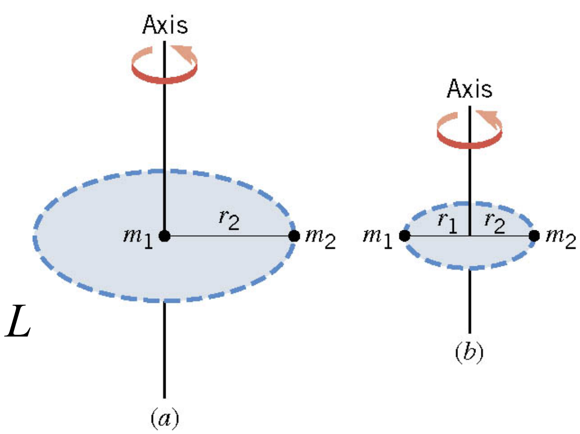
$$\sum \tau = I\,\alpha \qquad \boxed{I = \sum\left(mr^2\right)}$$

Requirement: Angular acceleration must be expressed in radians/s^2.

Moment of Inertia depends on axis of rotation.

Two particles each with mass, m, and are fixed at the ends of a thin rigid rod. The length of the rod is L. Find the moment of inertia when this object rotates relative to an axis that is perpendicular to the rod at (a) one end and (b) the center.

$$m_1 = m_2 = m$$

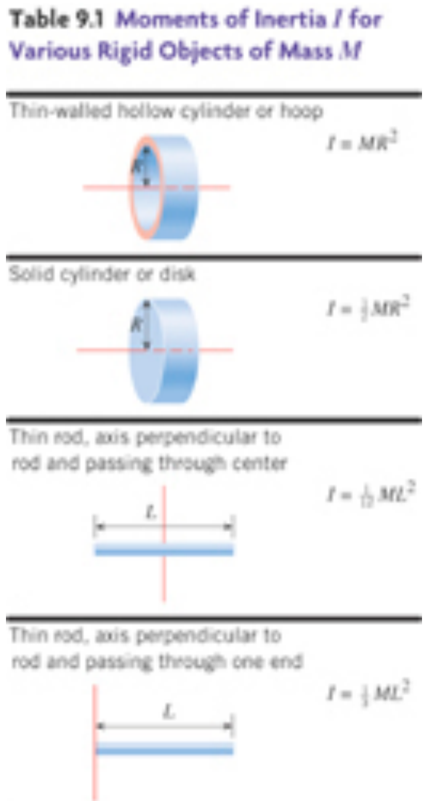


$$r_1 = 0, \quad r_2 = L \qquad\qquad r_1 = \frac{L}{2}, \quad r_2 = \frac{L}{2}$$

(a) $$I = \sum\left(mr^2\right) = m_1 r_1^2 + m_2 r_2^2 = m\left(0\right)^2 + m\left(L\right)^2$$
$$= mL^2$$

(b) $$I = \sum\left(mr^2\right) = m_1 r_1^2 + m_2 r_2^2 = m\left(L/2\right)^2 + m\left(L/2\right)^2$$
$$= \tfrac{1}{2} mL^2$$

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis

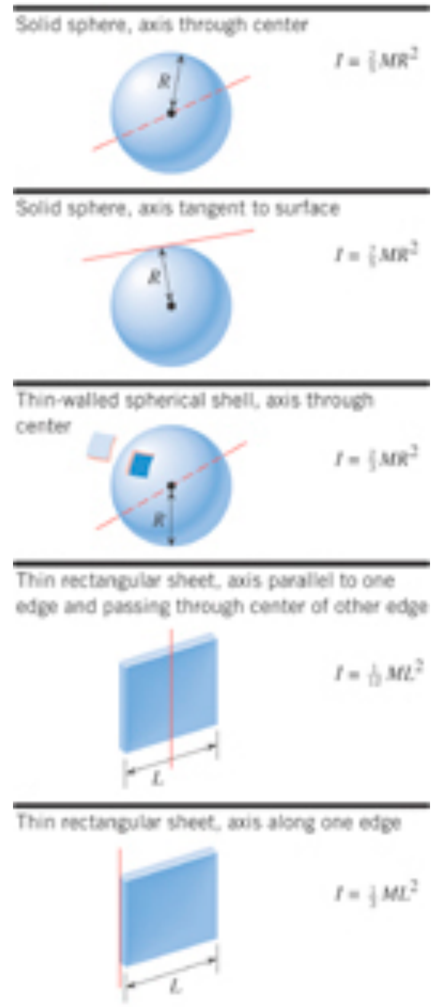


$$I = MR^2$$

$$I = \tfrac{1}{2}\,MR^2$$

$$I = \tfrac{1}{12}\,MR^2$$

$$I = \tfrac{1}{3}\,MR^2$$



$$I = \tfrac{2}{5}\,MR^2$$

$$I = \tfrac{7}{5}\,MR^2$$

$$I = \tfrac{2}{3}\,MR^2$$

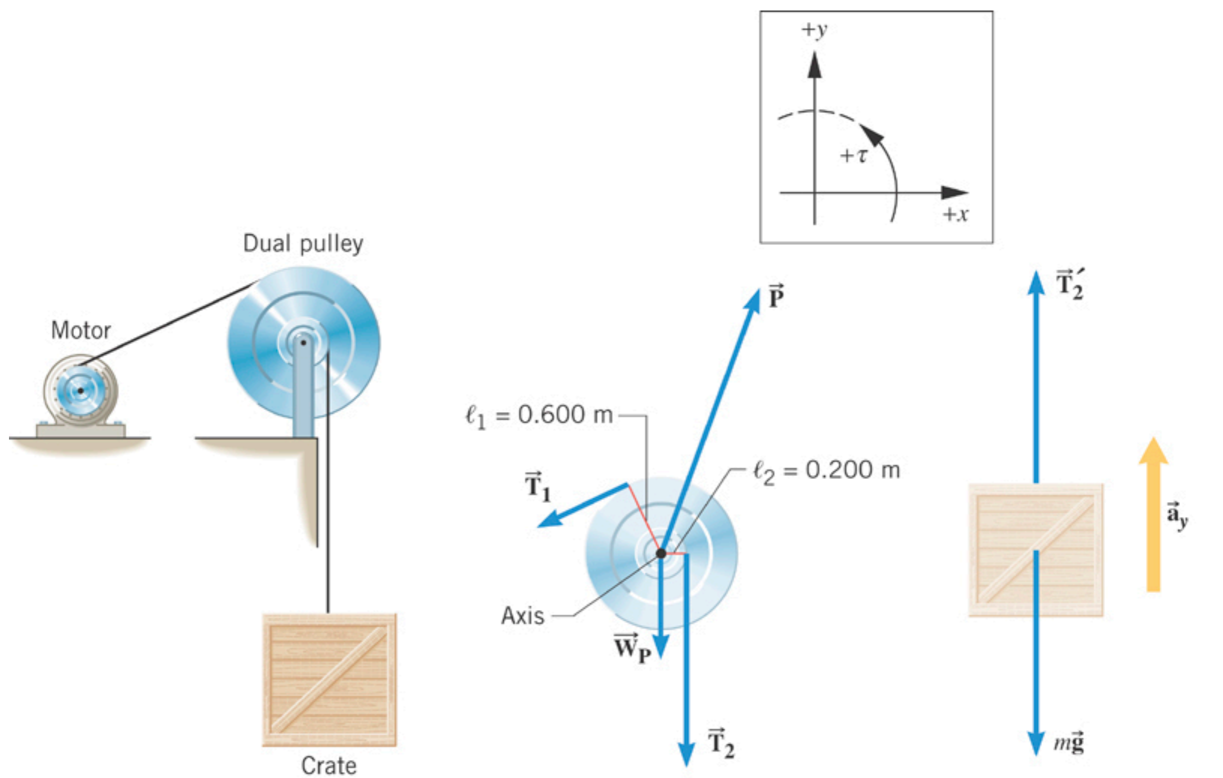
$$I = \tfrac{1}{12}\,MR^2$$

$$I = \tfrac{1}{3}\,MR^2$$

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis

Example 12 Hoisting a Crate

The combined moment of inertia of the dual pulley is 50.0 kg·m^2. The crate weighs 4420 N. A tension of 2150 N is maintained in the cable attached to the motor. Find the angular acceleration of the dual pulley.



(*a*) (*b*) Free-body diagram of pulley (*c*) Free-body diagram of crate

9.4 Newton's Second Law for Rotational Motion About a Fixed Axis

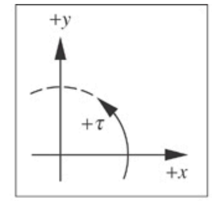
2nd law for linear motion of crate

$$\sum F_y = T_2 - mg = ma_y \qquad a_y = \ell_2 \alpha$$

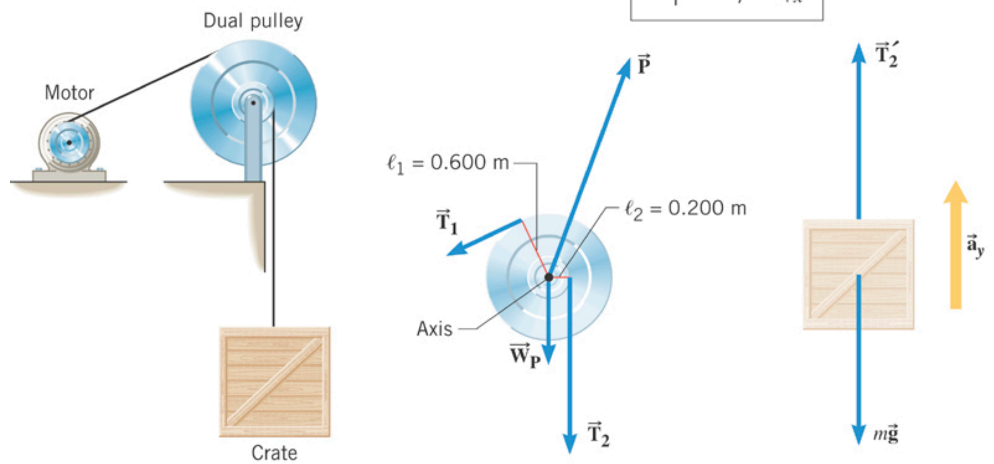
$$I = 46\,\mathrm{kg}\cdot\mathrm{m}^2$$
$$mg = 4420\ \mathrm{N}$$
$$T_1 = 2150\ \mathrm{N}$$

2nd law for rotation of the pulley

$$\sum \tau = T_1 \ell_1 - T_2 \ell_2 = I\alpha$$



$$T_2 - mg = m\ell_2 \alpha$$
$$T_1 \ell_1 - T_2 \ell_2 = I\alpha$$
$$T_2 = m\ell_2 \alpha + mg$$
$$T_2 = \frac{T_1 \ell_1 - I\alpha}{\ell_2}$$
$$m\ell_2^2 \alpha + I\alpha = T_1 \ell_1 - mg\ell_2$$
$$\alpha = \frac{T_1 \ell_1 - mg\ell_2}{m\ell_2^2 + I} = 6.3\ \mathrm{rad/s^2}$$



(*a*) (*b*) Free-body diagram of pulley (*c*) Free-body diagram of crate

$$\vec{\mathbf{T}}_2' = +T_2$$
$$\vec{\mathbf{T}}_2 = -T_2$$

$$T_2 = m\ell_2 \alpha + mg = \big(451(.6)(6.3) + 4420\big)\mathrm{N} = 6125\ \mathrm{N}$$

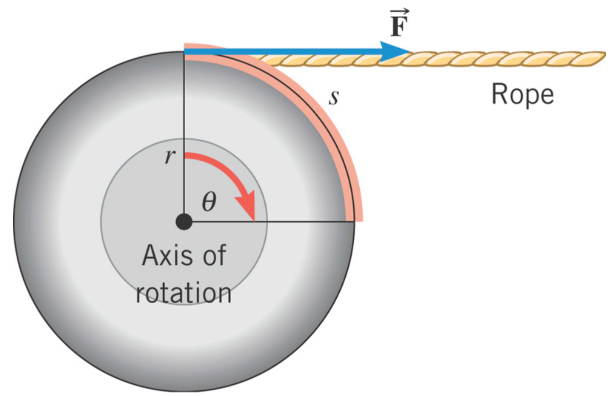
9.5 Rotational Work and Energy

Work rotating a mass:

$$W = Fs \qquad s = r\theta$$

$$= Fr\theta \qquad Fr = \tau$$

$$= \tau\theta$$



DEFINITION OF ROTATIONAL WORK

The rotational work done by a constant torque in turning an object through an angle is

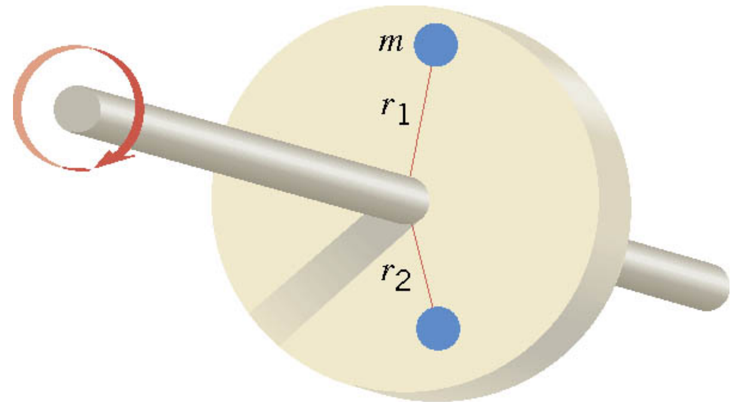
$$W_R = \tau\theta$$

Requirement: The angle must be expressed in radians.

SI Unit of Rotational Work: joule (J)

9.5 Rotational Work and Energy

Kinetic Energy of a rotating one point mass



$$KE = \tfrac{1}{2} m v_T^2 \qquad v_T = r\omega$$

$$= \tfrac{1}{2} m r^2 \omega^2$$

Kinetic Energy of many rotating point masses

$$KE = \sum\left(\tfrac{1}{2} m r^2 \omega^2\right) = \tfrac{1}{2}\left(\sum m r^2\right)\omega^2 = \tfrac{1}{2} I\omega^2$$

DEFINITION OF ROTATIONAL KINETIC ENERGY

The rotational kinetic energy of a rigid rotating object is

$$KE_R = \tfrac{1}{2} I\omega^2$$

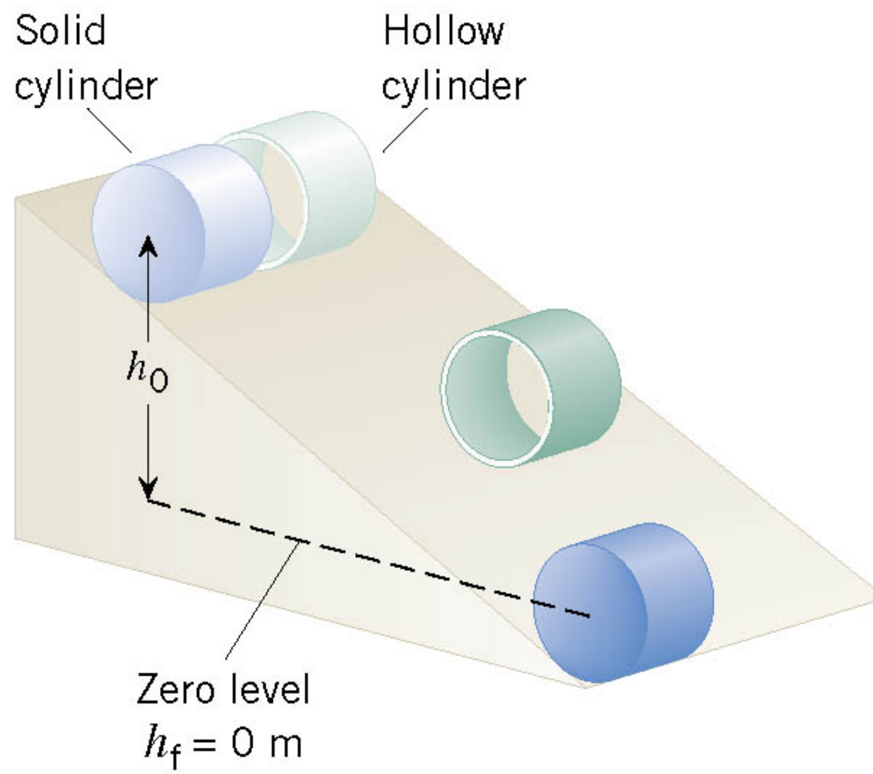
Requirement: The angular speed must be expressed in rad/s.

SI Unit of Rotational Kinetic Energy: joule (J)

Example 13 Rolling Cylinders

A thin-walled hollow cylinder (mass = m_h, radius = r_h) and a solid cylinder (mass = m_s, radius = r_s) start from rest at the top of an incline.

Determine which cylinder has the greatest translational speed upon reaching the bottom.



9.5 Rotational Work and Energy

Total Energy = Kinetic Energy + Rotational Energy + Potential Energy

$$E \quad = \quad \tfrac{1}{2}mv^2 \quad + \quad \tfrac{1}{2}I\omega^2 \quad + \quad mgh$$

ENERGY CONSERVATION $\quad E_f = E_i$

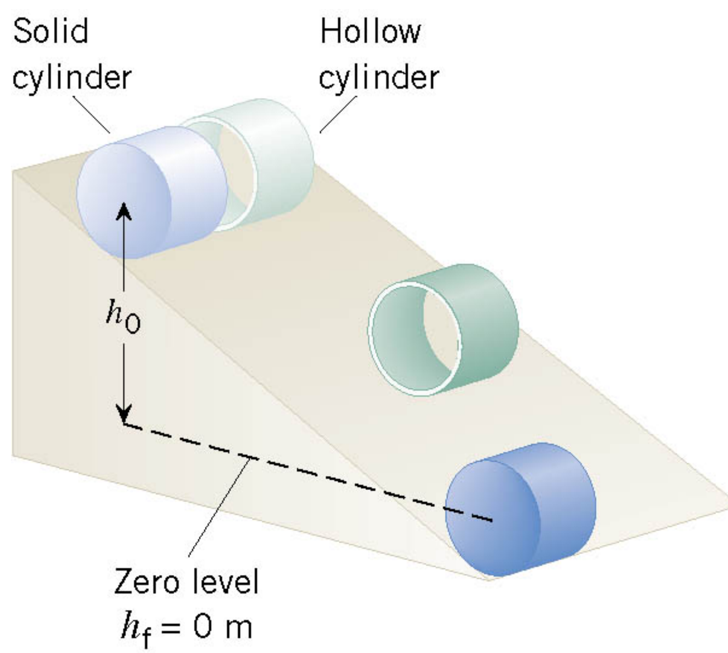
$$\tfrac{1}{2}mv_f^2 + \tfrac{1}{2}I\omega_f^2 + mgh_f = \tfrac{1}{2}mv_i^2 + \tfrac{1}{2}I\omega_i^2 + mgh_i$$

$$\boxed{\omega_f = v_f / r} \qquad 0 \qquad 0 \qquad 0$$

$$\tfrac{1}{2}mv_f^2 + \tfrac{1}{2}I\,v_f^2 / r^2 = mgh_i$$

$$\boxed{v_f = \sqrt{\frac{2mgh_o}{m + I/r^2}}}$$

Same mass for cylinder and hoop



The cylinder with the smaller moment of inertia will have a greater final translational speed.

DEFINITION OF ANGULAR MOMENTUM

The angular momentum *L* of a body rotating about a fixed axis is the product of the body's moment of inertia and its angular velocity with respect to that axis:

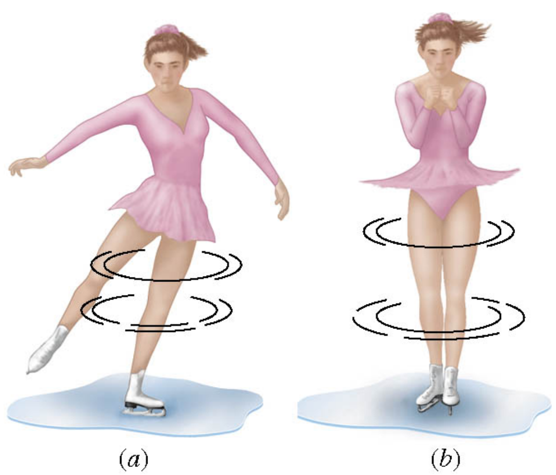
$$L = I\omega$$

Requirement: The angular speed must be expressed in rad/s.

SI Unit of Angular Momentum: kg·m^2/s

PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM

The angular momentum of a system remains constant (is conserved) if the net external torque acting on the system is zero.



Angular momentum, L

$$L_i = I_i\omega_i; \quad L_f = I_f\omega_f$$

No external torque

$\Rightarrow$ Angular momentum conserved

$$L_f = L_i$$

Moment of Inertia decreases

$$I = \sum mr^2,\ r_f < r_i$$

$$I_f < I_i$$

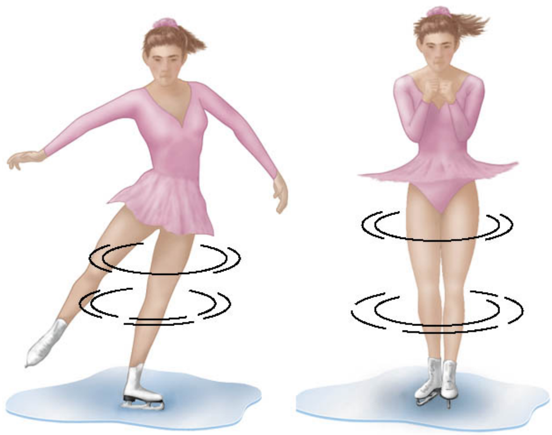
$$\frac{I_i}{I_f} > 1$$

$$I_f\omega_f = I_i\omega_i$$

$$\omega_f = \frac{I_i}{I_f}\omega_i; \quad \frac{I_i}{I_f} > 1$$

$\omega_f > \omega_i$ (angular speed increases)

9.6 Angular Momentum



From Angular Momentum Conservation

$$\omega_f = \left(I_i / I_f\right)\omega_i \qquad I_i / I_f > 1$$

Angular velocity increases

Is Energy conserved?

$$KE_f = \tfrac{1}{2} I_f \omega_f^2$$

$$= \tfrac{1}{2} I_f \left(I_i / I_f\right)^2 \omega_i^2$$

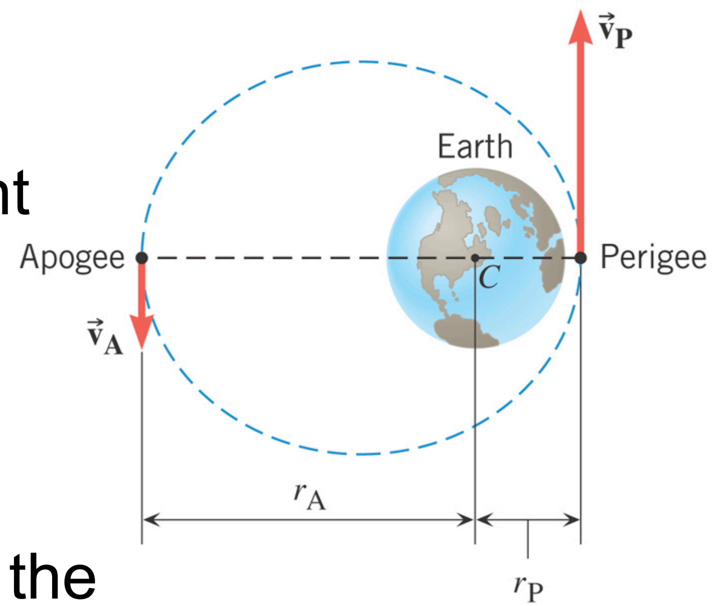
$$= \left(I_i / I_f\right)\left(\tfrac{1}{2} I_i \omega_i^2\right) \qquad KE_i = \tfrac{1}{2} I_i \omega_i^2;$$

$$= \left(I_i / I_f\right) KE_i \quad \Rightarrow \quad \text{Kinetic Energy increases}$$

Energy is not conserved because pulling in the arms does (NC) work on their mass and increases the kinetic energy of rotation

Example 15 A Satellite in an Elliptical Orbit

An artificial satellite is placed in an elliptical orbit about the earth. Its point of closest approach is 8.37x10^{6}m from the center of the earth, and its point of greatest distance is 25.1x10^{6}m from the center of the earth. The speed of the satellite at the perigee is 8450 m/s. Find the speed at the apogee.



Gravitational force along L (no torque)

Angular momentum conserved

$$I_A = mr_A^2; \quad I_P = mr_P^2$$

$$\omega_A = v_A/r_A; \quad \omega_P = v_P/r_P$$

$$I_A\omega_A = I_P\omega_P$$

$$mr_A^2\left(v_A/r_A\right) = mr_P^2\left(v_P/r_P\right)$$

$$r_A v_A = r_P v_P \quad \Rightarrow \quad v_A = \left(r_P/r_A\right)v_P = \left[(8.37\times10^6)/(25.1\times10^6)\right]\left(8450 \text{ m/s}\right) = 2820 \text{ m/s}$$