J. C. Maxwell   In 1864, Maxwell published a complete field theory of electromagnetism. One new, previously unknown phenomenon was necessary — Displacement Current

Recall Faraday's law of electromagnetic induction: $\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$

$$\Phi_B = \int_S \vec{B}\cdot d\vec{A}$$

Maxwell's displacement current

$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 i_{through} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

(Ampère's law)   (displacement current)

$$\Phi_E = \int_S \vec{E}\cdot d\vec{A}$$

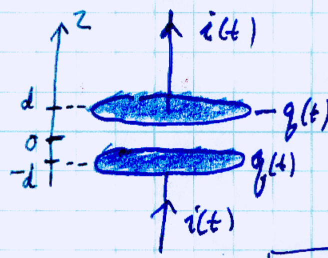
Define $i_D = \epsilon_0 \frac{d\Phi_E}{dt}$.

$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 (i + i_D)$$

charge current   displacement current

Displacement Current

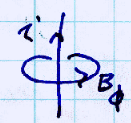
Example: Charging a Capacitor



$\frac{dq}{dt} = i$

What is the magnetic field, $\vec{B}(t)$?

Recall Ampère's Law ...



$\int \vec{B}\cdot d\vec{\ell} = \mu_0 i$

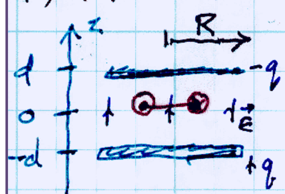
$\therefore \vec{B} = \frac{\mu_0 i}{2\pi r}\hat{\phi}$

(•) For $z > d$ or $z < -d$,

$$B_\phi(t)\, 2\pi r = \mu_0 (i + i_D) \approx \mu_0 i$$

$$B_\phi(t) \approx \frac{\mu_0 i(t)}{2\pi r} \quad \text{(neglecting radiation)}$$

(•) For $-d < z < d$,



$B_\phi \cdot 2\pi r = \mu_0 (i + i_D) = \mu_0 i_D$

$i_D = \epsilon_0 \frac{dE_z}{dt} \pi r^2$ (or πR^2 for $r>R$)

Gauss's Law: $E_z = \frac{q/\epsilon_0}{\pi R^2}$

$\therefore i_D = \dot{q} \frac{r^2}{R^2}$ (or $\dot{q}$ for $r>R$)

Thus,

$$B_\phi(t) = \begin{cases} \frac{\mu_0 i(t) r}{2\pi R^2} & \leftarrow \text{for } r<R \\ \frac{\mu_0 i(t)}{2\pi r} & \leftarrow \text{for } r>R \end{cases}$$

Maxwell's Equations

Maxwell's Equations

(•) $\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} q_{enclosed}$ Gauss's Law $\nabla\cdot\vec{E} = \rho/\epsilon_0$

(•) $\oint_S \vec{B}\cdot d\vec{A} = 0$ Gauss's Law for magnetism $\nabla\cdot\vec{B} = 0$

(•) $\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A}$ Faraday's Law $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$EMF = -\frac{d\Phi_B}{dt}$

(•) $\oint_C \vec{B}\cdot d\vec{l} = \mu_0 i_{thru} + \mu_0\epsilon_0\frac{d}{dt}\int_S \vec{E}\cdot d\vec{A}$ Ampere-Maxwell Law $\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

$= \mu_0(i + i_D)$ where $i_D = \epsilon_0\frac{d\Phi_E}{dt}$

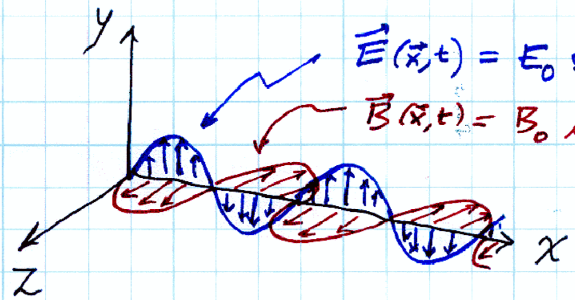
the differential form of maxwell's field equations.

Also, $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$.

Terms that are zero in "vacuum", i.e. where there is no matter.

Electromagnetic Waves

Electromagnetic Waves



$$\vec{E}(\vec{x},t) = E_0 \sin(kx-\omega t)\,\hat{j}$$
$$\vec{B}(\vec{x},t) = B_0 \sin(kx-\omega t)\,\hat{k}$$

This is a harmonic, polarized, plane wave.

We'll show that it satisfies the four Maxwell equations.

(•) $\nabla\cdot\vec{E} = \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z}$
$= 0$ ✓

(•) $\nabla\cdot\vec{B} = \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} + \dfrac{\partial B_z}{\partial z}$
$= 0$ ✓

(•) $\nabla\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = \hat{k}\,\dfrac{\partial E_y}{\partial x}$

$= E_0 k \cos(kx-\omega t)\,\hat{k}$

$= -\dfrac{\partial \vec{B}}{\partial t} = B_0\,\omega \cos(kx-\omega t)\,\hat{k}$

$\therefore\ \boxed{kE_0 = \omega B_0}$ ✓

(•) $\nabla\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & B_y & B_z \end{vmatrix} = -\hat{j}\,\dfrac{\partial B_z}{\partial x}$

$= -B_0 k \cos(kx-\omega t)\,\hat{j}$

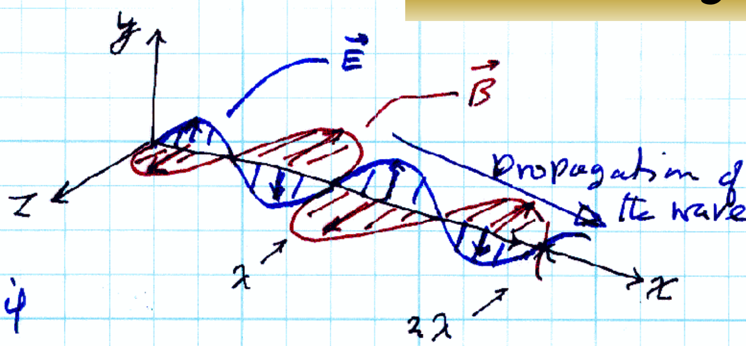
$= \mu_0\epsilon_0 \dfrac{\partial \vec{E}}{\partial t} = -\mu_0\epsilon_0 \cancel{E_0}\,\omega \cos(kx-\omega t)\,\hat{j}$

$\therefore\ \boxed{kB_0 = \mu_0\epsilon_0\,\omega E_0}$ ✓

Electromagnetic Waves

Electromagnetic Waves

$$\vec{E}(\vec{x},t) = E_0 \sin(kx-\omega t)\,\hat{j}$$
$$\vec{B}(\vec{x},t) = B_0 \sin(kx-\omega t)\,\hat{k}$$



Maxwell's equations are satisfied if

$$kE_0 = \omega B_0 \quad \text{and} \quad kB_0 = \mu_0 \epsilon_0 \omega E_0 \quad (*)$$

Wavelength (λ), frequency (f), wave speed (c)

- $\lambda = \dfrac{2\pi}{k}$ because $\sin[k(x+\lambda)-\omega t]$
$= \sin(kx-\omega t+2\pi)$
$= \sin(kx-\omega t)$

- $T = \dfrac{2\pi}{\omega}$ because $\sin[kx-\omega(t+T)]$
$= \sin[kx-\omega t \mp 2\pi]$
$= \sin(kx-\omega t)$

$f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$

- $c = \dfrac{\lambda}{T} = \lambda f = \dfrac{\omega}{k}$

But $(*)$ implies $\dfrac{\omega}{k} = \dfrac{E_0}{B_0} = \dfrac{k}{\mu_0 \epsilon_0 \omega}$

$$\therefore \quad \boxed{\frac{\omega^2}{k^2} = \frac{1}{\mu_0 \epsilon_0} \quad \text{or} \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}}$$

The wave speed

$c = 2.998 \times 10^8 \ \frac{m}{s}$

= the speed of all electromagnetic waves in vacuum, including visible light.

c is the same for all λ.

Energy Transport in Electromagnetic Fields

Energy and Momentum Transport in electromagnetic waves

(a) Poynting Vector. We define

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

Theorem: $\vec{S}$ is the energy flux vector.

I.e., $\vec{S}$ = energy per unit time per unit area

$[W/m^2]$

Proof

$$\nabla \cdot \vec{S} = \frac{1}{\mu_0} \nabla \cdot (\vec{E} \times \vec{B})$$

$$= \frac{1}{\mu_0} \{ (\nabla \times \vec{E}) \cdot \vec{B} - \vec{E} \cdot (\nabla \times \vec{B}) \}$$

$$= \frac{1}{\mu_0} \{ -\frac{\partial \vec{B}}{\partial t} \cdot \vec{B} - \vec{E} \cdot \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \}$$

$$= -\frac{\partial}{\partial t} \{ \frac{1}{2\mu_0} B^2 + \frac{\epsilon_0}{2} E^2 \}$$

$$= -\frac{\partial}{\partial t} \{ u_B + u_E \}$$

QED

Example: For the polarized plane wave,

$$\vec{E} = E_0 \cos(kx - \omega t)\, \hat{j}$$

$$\vec{B} = B_0 \cos(kx - \omega t)\, \hat{k}$$

$$\vec{S} = \frac{1}{\mu_0} E_0 B_0 \cos^2(kx - \omega t)\, \hat{i}$$

$$S_{avg} = \frac{E_0 B_0}{\mu_0} \cdot \frac{1}{2} = \frac{1}{\mu_0} E_{RMS} B_{RMS}$$

$$S_{avg} = \frac{1}{\mu_0 c} E_{RMS}^2 = c\epsilon_0 E_{RMS}^2$$

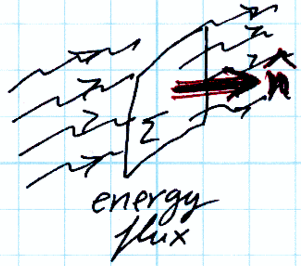
Energy flows in the direction of propagation ($\hat{i}$);

Intensity $= S_{avg}$ ← mean energy flux

Momentum Transport and Radiation Pressure

Momentum Transfer and Radiation Pressure

Through a surface Σ,



$$\frac{dU}{dt} = \int_{\Sigma} \vec{S} \cdot \hat{n} \; dA$$

$$\left(\frac{dU}{dt}\right)_{avg} = \int_{\Sigma} \vec{S}_{avg} \cdot \hat{n} \, dA \leftarrow [\text{watts}]$$

(•) Also, the momentum flow is

$$\frac{d\vec{P}}{dt} = \frac{1}{c} \frac{dU}{dt} \hat{z} \qquad (\hat{z} = \text{propagation direction})$$

$$\left(\frac{d\vec{P}}{dt}\right)_{avg} = \frac{1}{c} S_{avg} A \; \hat{z} = \frac{I}{c} A \; \hat{z}$$

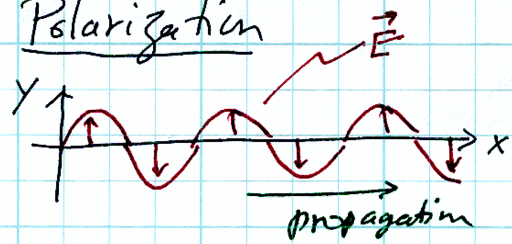
units $\frac{1}{m/s} \; \frac{J}{s\,m^2} \, m^2 = \frac{J}{m} = N$

Absorbent Surface $F(\text{on } \Sigma) = \frac{I}{c} A$

Reflective Surface $F(\text{on } \Sigma) = \frac{2I}{c} A$

(•) Radiation pressure $= F/A = \frac{I}{c}$ or $\frac{2I}{c}$

Polarization

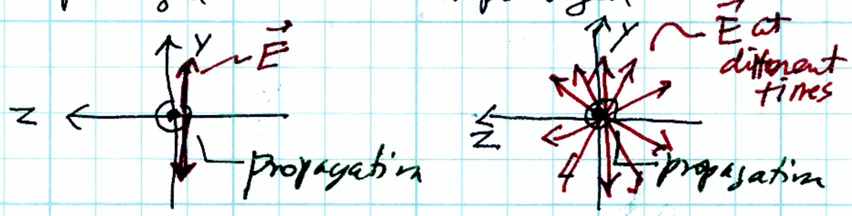


= an e.m. wave that is polarized in the y direction, and propagating in the x direction.

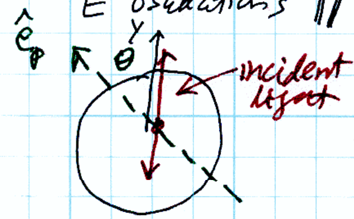
We use the direction of $\vec{E}$ to define the direction of polarization.

Most light is unpolarized.

polarized   unpolarized



A polarizer absorbs all $\vec{E}$ field oscillations $\perp$ to the polarizer axis, leaving only the $\vec{E}$ oscillations $\parallel$ to the polarizer axis.



incident : $\vec{E} = E_0 \hat{j}$

transmitted : $\vec{E} = E_0 \cos\theta \; \hat{e}_p$

Intensities $\quad I \propto E^2$

(•) $I_{out} = I_{inc} \cos^2\theta$ polarized incident light

(•) $I_{out} = 0.5 \, I_{inc}$ unpolarized incident light (avg. $\cos^2\theta = 0.5$)

Polarization and Polarizers