Orbits of the Planets - Celestial Mechanics

The earliest days of the Scientific Revolution.

Recall the history...

Copernicus (1473 - 1543)

Galileo (1564 - 1642)

Kepler (1571 - 1630)

Newton (1642 - 1727)

(Newton explained why)

Johannes Kepler

He was the Imperial Mathematician of the Holy Roman Emperor; worked in Prague.

After 20 years of calculations, based on night-sky observations made by his predecessor Tycho Brahe, Kepler summarized the motion of the solar system:

Three Laws of Planetary Orbits

/1/ A planet moves on an elliptical orbit, with the sun at one focal point.

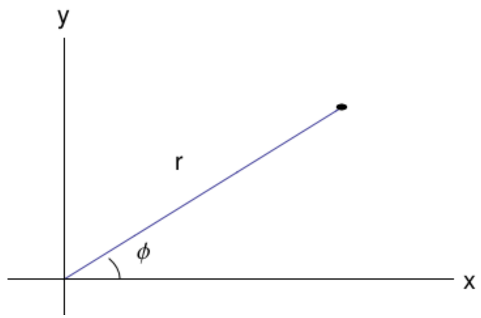
/2/ The radial vector sweeps out equal areas in equal times.

/3/ The square of the period is proportional to the cube of the semi-major axis; T^2 /a^3 is constant.

Ellipse Geometry

Now we need to describe an ellipse with the origin at one focal point.

The usual way is to use *polar coordinates.*

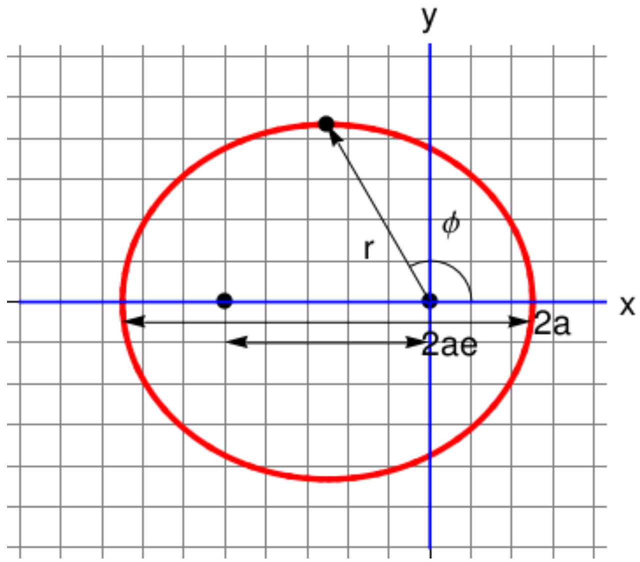


$$x = r \cos \{phi\}$$

$$y = r \sin \{phi\}$$

$$r = (x^2 + y^2)^{½}$$

$$\tan \{phi\} = y/x$$



An ellipse must be defined by 2 parameters.

- The semimajor axis = a

= one-half of the large diameter.

- The eccentricity = e

= (distance between the focal points) divided by (large diameter).

- For planetary orbits we also use …

Perihelion = a (1 - e)

Aphelion = a (1 + e)

$$r = \frac{a\,(1-e^2)}{1 + e \cos \{phi\}}$$

Newton’s calculations for a circular orbit

The simplest case is a circular orbit.

(A circle is an ellipse with e = 0.)

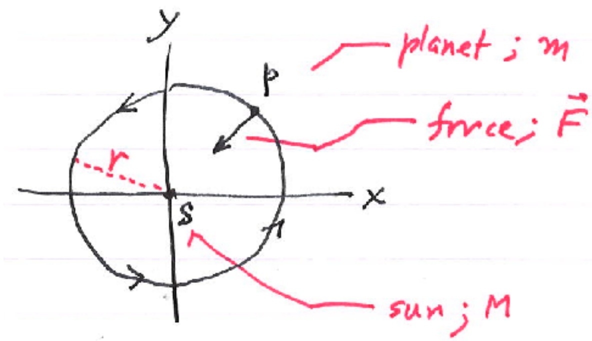
So let’s calculate the properties of circular orbits. The sun is at the origin of the coordinate system; solar mass = M. The planet moves on a circle; speed is constant.

Following Kepler, we should have T^2/r^3 = constant.

That will be true if $f(r) = 1/r^2$.

So Newton concluded that the force of gravity must be an inverse square law: F inversely proportional to r^2.

Then T = 2 {pi} $(r^3 / GM)^{½}$.



$$\vec{F} = GMm\, f(r)\, (-\hat{r})$$

$$\vec{a} = \frac{v^2}{r}\, (-\hat{r})$$

Newton's 2nd law ⇒ $GMm\, f(r) = \frac{mv^2}{r}$

$$\therefore \quad v = \sqrt{GM\, r\, f(r)}$$

The period of revolution

$$T = \frac{2\pi r}{v} = 2\pi \sqrt{\frac{r}{GM\, f(r)}}$$

For elliptical orbits, T = 2 {pi} $(a^3 / GM)^{½}$

Parametric equations for an elliptical orbit

A fairly involved calculation from Newton's laws gives the equations for the motion of a planet or satellite.

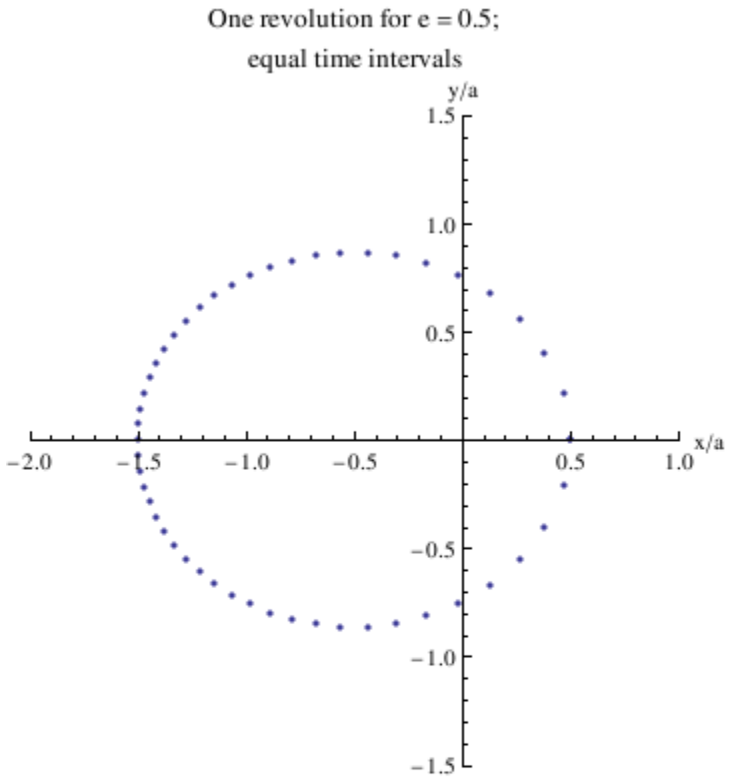
As functions of an independent variable ψ, the polar coordinates and time are

$$r(\psi) = a\,(1 - e\,cos(\psi)\,)$$

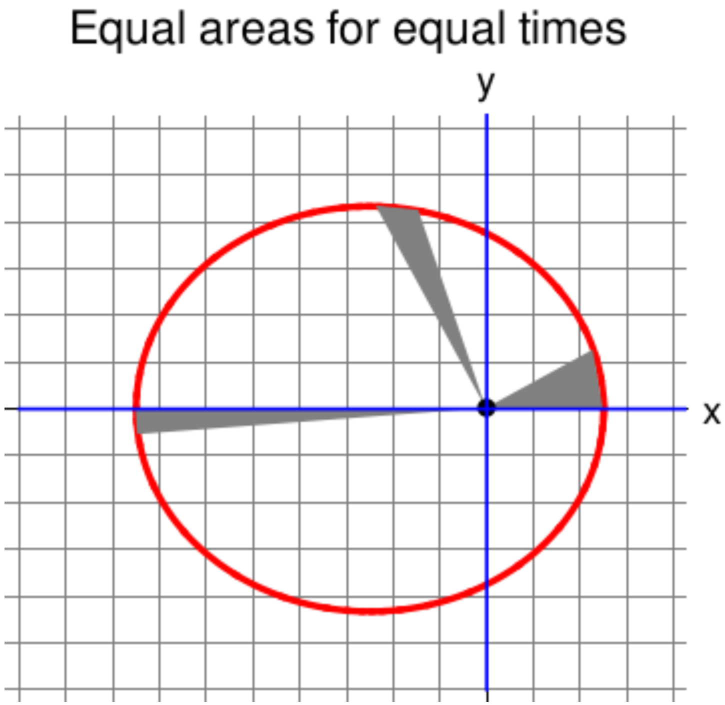
$$tan\,(\phi/2) \;=\; \sqrt{(1-e)/(1+e)}\;\; tan\,(\psi/2)$$

$$t(\psi) \;=\; T/(2\pi)\,(\,\psi - e\,sin(\psi)\,)$$

Graphical analysis



Kepler’s second law (“the law of equal areas”) is a consequence of conservation of angular momentum.

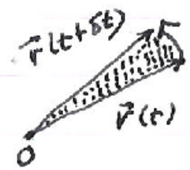


The definition of *angular momentum* is

L = r x p *(these are vectors!)*

For the orbit, L = m r $v_{\{phi\}}$ *(magnitude)*

Consider the area swept out in a small time interval δt



$$\delta A \approx \frac{1}{2} r\, v_\phi\, \delta t$$

$$\frac{\delta A}{\delta t} = \frac{r v_\phi}{2} = \frac{L}{2m}$$

(limit $\delta t \to 0$)

Result: dA/dt is constant because L is constant.

Therefore the radial vector sweeps out equal areas in equal times.

Q.E.D.

Example : Halley's comet

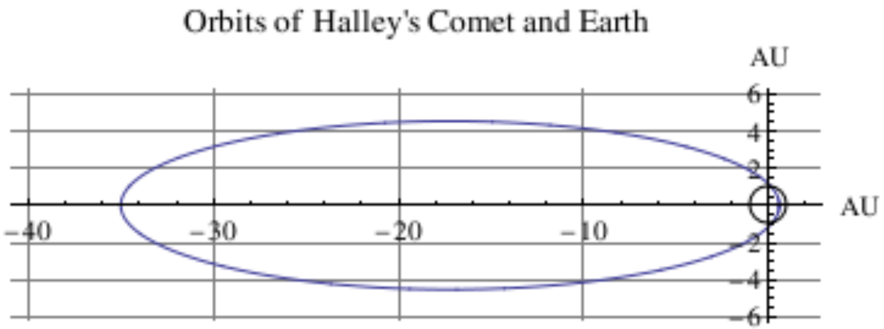
Edmund Halley and Isaac Newton

From Newton's theory, Halley showed that some known comets were repeated viewings of the same objects, which revolve around the sun with large eccentricity.

For example, Halley's Comet:

$T = 75.3$ years

$e = 0.967$



1 AU = 149.6 x 10^9 m = the mean distance from the sun to the earth

Calculate the semimajor axis from the period of revolution.

$$T^2 = 2 \{pi\} (a^3 /GM)^{1/2}$$

Easier, use this trick:

$$(T^2/a^3)_{HC} = (T^2/a^3)_{Earth} = 1\ y^2/AU^3$$

$$a_{HC} = (T_{HC}/1y)^{2/3}\ AU$$