The Solar System

VIDEO LECTURES: 6-1 6-2 6-3

1 KEPLER'S LAWS OF PLANETARY MOTION

Kepler's first law is that the planets travel on ellipses with the sun at one focal point. Newton deduced from this empirical observation that the gravitational force on the planet must be proportional to $1/r^2$ where r is the distance from the sun.

Figure 1 shows a possible planetary orbit. The ellipse is characterized by two parameters: $a =$ semimajor axis and $e =$ eccentricity.

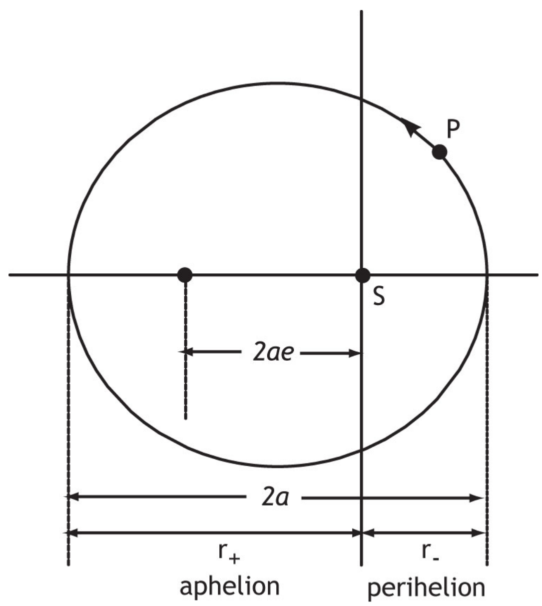

Figure 1: A possible planetary orbit. The sun S is at the origin, which is one focal point of the ellipse, and the planet P moves on the ellipse. The large diameter is $2a$, where a is called the semimajor axis. The distance between the foci is $2ae$ where e is called the eccentricity. The perihelion distance is $r_- = a(1-e)$ and the aphelion distance is $r_+ = a(1+e)$. A circle is an ellipse with $e = 0$.

1.1 Kepler's third law

Kepler's third law relates the period T and the semimajor axis a of the ellipse. To the accuracy of the data available in his time, Kepler found that T^2 is proportional to a^3. The next example derives this result from Newtonian mechanics, for the special case of a circular orbit. A circle is an ellipse with eccentricity equal to zero; then the semimajor axis is the radius.

Example 1-1. Show that $T^2 \propto r^3$ for a planet that revolves around the sun on a circular orbit of radius r.[1]

Solution. In analyzing the problem, we will neglect the motion of the sun. More precisely, both the sun and the planet revolve around their center of mass. But because the sun is much more massive than the planet, the center of mass is approximately at the position of the sun, so that the sun may be considered to be at rest. Neglecting the motion of the sun is a good approximation. A more accurate calculation is in Exercise 8.

Let m denote the mass of the planet, and M the mass of the sun.

For a circular orbit the angular speed of the planet is constant, so $d\omega/dt = 0$. Therefore the acceleration is $\mathbf{a} = -r\omega^2\widehat{\mathbf{r}}$; or, in terms of the speed $v = r\omega$,

$$\mathbf{a} = -\frac{v^2}{r}\widehat{\mathbf{r}}. \tag{1}$$

The direction is $-\widehat{\mathbf{r}}$, i.e., centripetal, toward the sun. The gravitational force exerted by the sun on the planet is

$$\mathbf{F} = -\frac{GMm}{r^2}\widehat{\mathbf{r}}, \tag{2}$$

which is also centripetal. Equation (2) is Newton's theory of Universal Gravitation, in which the force is proportional to $1/r^2$.

Newton's second law of motion states that $\mathbf{F} = m\mathbf{a}$. Therefore,

$$\frac{mv^2}{r} = \frac{GMm}{r^2}. \tag{3}$$

The speed of the planet is

$$v = \left(\frac{GM}{r}\right)^{1/2} . \tag{4}$$

The distance traveled in time T (one period of revolution) is $2\pi r$ (the circumference of the orbit), so the speed is $v = 2\pi r/T$. Substituting this expression for v into (4) gives

$$\left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}. \tag{5}$$

Or, rearranging the equation,

$$T^2 = \frac{4\pi^2 r^3}{GM}; \tag{6}$$

[1]We consider an ideal case in which the *other* planets have a negligible effect on the planet being considered. This is a good approximation for the solar system, but not exact.

we see that T^2 is proportional to r^3, as claimed.

In obtaining (6) we neglected the small motion of the sun around the center-of-mass point. This is a very good approximation for the solar system. In this approximation, T^2/r^3 is constant; i.e., T^2/r^3 *has the same value for all nine planets.*

We have only considered a special case—a circular orbit. In general, a planetary orbit is an *ellipse*. The calculation of elliptical orbits is more complicated, but the final result for the period is simple

$$T^2 = \frac{4\pi^2 a^3}{G(M+m)} \approx \frac{4\pi^2 a^3}{GM} \tag{7}$$

where a is the semimajor axis.

1.2 Kepler's second law

Kepler's second law states that the radial vector sweeps out equal areas in equal times. This law is illustrated in Fig. 2. In Newtonian mechanics it is a consequence of conservation of angular momentum. The next two examples show how Kepler's second law follows from Newton's theory.

Example 1-2. **Conservation of angular momentum**

The angular momentum L of an object of mass m that moves in the xy plane is defined by

$$L = m\left(xv_y - yv_x\right). \tag{8}$$

Show that L is constant if the force on the object is central.

Solution. To show that a function is constant, we must show that its derivative is 0.[2] In (8), the coordinates x and y, and velocity components v_x and v_y, are all functions of time t. But the particular combination in L is constant, as we now show. The derivative of L is

$$\begin{aligned}\frac{dL}{dt} &= m\left[\frac{dx}{dt}v_y + x\frac{dv_y}{dt} - \frac{dy}{dt}v_x - y\frac{dv_x}{dt}\right] \\ &= mv_xv_y + xF_y - mv_yv_x - yF_x \\ &= xF_y - yF_x. \end{aligned} \tag{9}$$

In the first step we have used the fact that $d\mathbf{x}/dt = \mathbf{v}$, and $d\mathbf{v}/dt = \mathbf{a}$; also, by Newton's second law, the acceleration $\mathbf{a}$ is equal to $\mathbf{F}/m$. The final line (9) is called the *torque* on the object.

For any central force the torque is 0. The term "central force" means that the force is in the direction of $\pm\hat{\mathbf{r}}$, i.e., along the line to the origin. (The sign—attractive

[2] The derivative of any constant is 0.

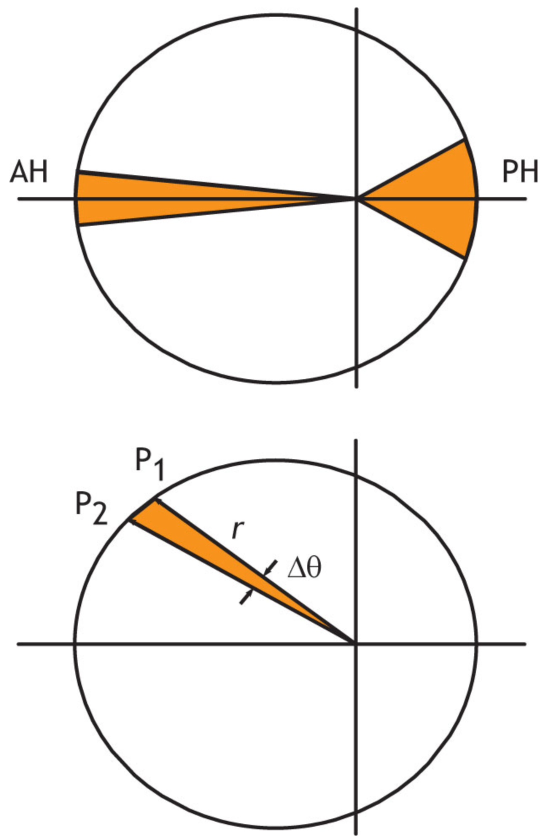

Figure 2: **Kepler's second law.** The radial vector sweeps out equal areas in equal times. (a) The radial vector sweeps out the shaded region as the planet moves from P_1 to P_2. (b) The planet moves faster near perihelion (PH) and slower near aphelion (AH).

or repulsive— is unimportant for the proof of conservation of angular momentum.) Figure 3 shows a central force **F** toward the origin. The components of **F** are

$$F_x = -F\cos\theta \quad \text{and} \quad F_y = -F\sin\theta \tag{10}$$

where F is the strength of the force and the minus signs mean that **F** is toward 0. Thus the torque on the object is

$$\begin{aligned} \text{torque} &= xF_y - yF_x \\ &= -r\cos\theta\,F\sin\theta + r\sin\theta\,F\cos\theta = 0. \end{aligned} \tag{11}$$

Since the torque is 0, equation (9) implies that $dL/dt = 0$. Since the derivative is 0, the angular momentum L is constant, as claimed.

Example 1-3. Kepler's law of equal areas

Show that the radial vector from the sun to a planet sweeps out equal areas in equal times.

Solution. Figure 2(a) shows the elliptical orbit. The shaded area ΔA is the area swept out by the radial vector between times t and $t + \Delta t$. The shaded area may be approximated by a triangle, with base r and height $r\Delta\theta$, where $\Delta\theta$ is the change of the angular position between t and $t + \Delta t$. Approximating the area as a triangle

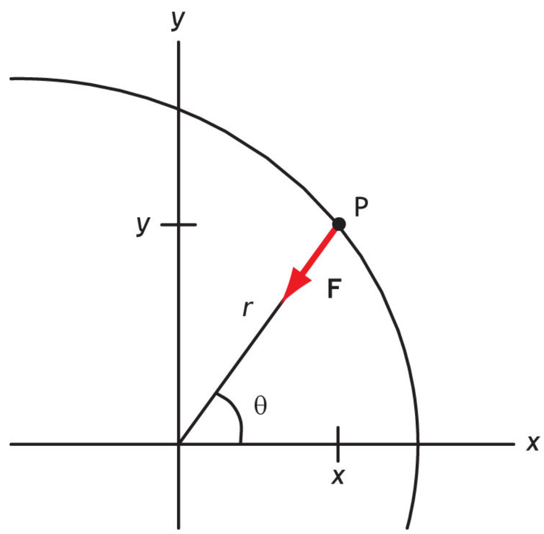

Figure 3: An attractive central force. The Cartesian coordinates at P are $x = r\cos\theta$ and $y = r\sin\theta$. The force components are $F_x = -F\cos\theta$ and $F_y = -F\sin\theta$ where F is the magnitude of the force vector. The torque, $xF_y - yF_x$, is 0.

is a good approximation for small Δt. Now consider the limit $\Delta t \to 0$; i.e., Δt and ΔA become the differentials dt and dA. The area of the triangle becomes

$$dA = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times r \times r d\theta = \tfrac{1}{2} r^2 d\theta. \tag{12}$$

Thus, in the limit $\Delta t \to 0$, where we replace Δt by dt,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \tfrac{1}{2} r^2 \omega. \tag{13}$$

We'll use this result presently.

But now we must express the angular momentum in polar coordinates. The position vector of M is $\mathbf{x} = r\widehat{\mathbf{r}}$, and its x and y components are

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta. \tag{14}$$

The velocity vector is

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{dr}{dt}\widehat{\mathbf{r}} + r\frac{d\widehat{\mathbf{r}}}{dt} = \frac{dr}{dt}\widehat{\mathbf{r}} + r\frac{d\theta}{dt}\widehat{\boldsymbol{\theta}}; \tag{15}$$

note that $d\widehat{\mathbf{r}} = \widehat{\boldsymbol{\theta}}\, d\theta$.[3] So, the x and y components of velocity are

$$v_x = \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta, \tag{16}$$

$$v_y = \frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta. \tag{17}$$

[3] Exercise 3.

Now, L is defined in (8); substituting the polar expressions for x, y, v_x and v_y we find

$$\begin{aligned} L &= m\left(xv_y - yv_x\right) \\ &= m\left[r\frac{dr}{dt}\cos\theta\sin\theta + r^2\frac{d\theta}{dt}\cos^2\theta\right] \\ &\quad -m\left[r\frac{dr}{dt}\sin\theta\cos\theta - r^2\frac{d\theta}{dt}\sin^2\theta\right] \\ &= mr^2\frac{d\theta}{dt}\left(\cos^2\theta + \sin^2\theta\right) = mr^2\frac{d\theta}{dt}. \end{aligned} \tag{18}$$

The result is

$$L = mr^2\omega. \tag{19}$$

Comparing this result to (13) we see that

$$\frac{dA}{dt} = \frac{L}{2m}. \tag{20}$$

But L is a constant of the motion by conservation of angular momentum. Thus dA/dt is constant. In words, *the rate of change of the area is constant*, i.e., independent of position on the orbit. Hence Kepler's second law is explained: The area increases at a constant rate, so *equal areas are swept out in equal times.*

2 THE INVERSE SQUARE LAW

Kepler's first law is that a planet travels on an ellipse with the sun at one focal point. We will prove that this observation implies that the force on the planet must be an inverse square law, i.e., proportional to $1/r^2$ where r is the distance from the sun. The calculations depend on all that we have learned about derivatives and differentiation.

The equation for an elliptical orbit in polar coordinates (r, θ) is

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta} \tag{21}$$

where a = semimajor axis and e = eccentricity. Figure 1 shows a graph of the ellipse. What force is implied by the orbit equation (21)? The radial acceleration is[4]

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2. \tag{22}$$

The first term involves the change of radius; the second term is the centripetal acceleration $-r\omega^2$. Now, a_r must equal F_r/m by Newton's second law. To determine the radial force F_r we must express a_r as a function of r. We know that angular momentum is constant; by (19),

$$mr^2\frac{d\theta}{dt} = L, \quad \text{so} \quad \frac{d\theta}{dt} = \frac{L}{mr^2}. \tag{23}$$

Now starting from (21), and applying the chain rule,[5]

$$\begin{aligned}\frac{dr}{dt} &= \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{-a(1-e^2)}{(1+e\cos\theta)^2}\left(-e\sin\theta\right)\frac{d\theta}{dt} \\ &= \frac{a(1-e^2)e\sin\theta}{(1+e\cos\theta)^2}\,\frac{L(1+e\cos\theta)^2}{ma^2(1-e^2)^2} = \frac{Le\sin\theta}{ma(1-e^2)};\end{aligned} \tag{24}$$

and, taking another derivative,

$$\frac{d^2r}{dt^2} = \frac{Le\cos\theta}{ma(1-e^2)}\,\frac{d\theta}{dt} = \frac{Le\cos\theta}{ma(1-e^2)}\,\frac{L}{mr^2}. \tag{25}$$

Combining these results in (22), the radial component of the acceleration is

$$\begin{aligned}a_r &= \frac{L^2e\cos\theta}{m^2a(1-e^2)r^2} - r\left(\frac{L}{mr^2}\right)^2 \\ &= \frac{L^2}{m^2r^2}\left\{\frac{e\cos\theta}{a(1-e^2)} - \frac{1+e\cos\theta}{a(1-e^2)}\right\} = \frac{-L^2}{m^2a(1-e^2)r^2}.\end{aligned} \tag{26}$$

[4] See Exercise 4.

[5] The calculations of (24) and (25) require these results from calculus: the derivative (with respect to θ) of $\cos\theta$ is $-\sin\theta$, and the derivative of $\sin\theta$ is $\cos\theta$.

By Newton's second law, then, the radial force must be

$$F_r = ma_r = -\frac{k}{r^2} \quad \text{where} \quad k = \frac{L^2}{ma(1-e^2)}. \tag{27}$$

Our result is that the force on the planet must be an attactive inverse-square-law, $F_r = -k/r^2$. The orbit parameters are related to the force parameter k by

$$L^2 = ma(1-e^2)k. \tag{28}$$

Newton's Theory of Universal Gravitation

From the fact that planetary orbits are elliptical, Newton deduced that $F_r = -k/r^2$. Also, k must be proportional to the planet's mass m because $T^2 \propto a^3$, independent of the mass. But then k must also be proportional to the solar mass, because for every action there is an equal but opposite reaction. Therefore the force vector must be

$$\mathbf{F} = F_r\widehat{\mathbf{r}} = -\frac{GMm}{r^2}\widehat{\mathbf{r}} \tag{29}$$

where G is a universal constant. Newton's theory of universal gravitation states that any two masses in the universe, m and M, attract each other according to the force (29).

Newton's gravitational constant G cannot be determined by astronomical observations, because the solar mass M is not known independently. G must be measured in the laboratory. An accurate measurement of G is very difficult, and was not accomplished in the time of Newton. The first measurement of G was by Henry Cavendish in 1798. G is hard to measure because gravity is extremely weak,

$$G = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{s}^{-2}\,\mathrm{kg}^{-1}. \tag{30}$$

Newton's theory of gravity is very accurate, but not exact. A more accurate theory of gravity—the theory of general relativity—was developed by Einstein. In relativity, planetary orbits are not perfect ellipses; the orbits *precess* very slowly. Indeed this precession is observed in precise measurements of planetary positions, and the measurements agree with the relativistic calculation.

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The examples in this introduction to classical mechanics show how calculus is used to understand profound physical observations such as the motion of the planets. Calculus is essential in the study of motion.

EXERCISES

1. The Earth's orbit around the Sun is nearly circular, with radius $R_\oplus = 1.496 \times 10^{11}$ m and period $T = 1$ y. From this, and the *laboratory* measurement of Newton's gravitational constant, $G = 6.67 \times 10^{-11}\ \mathrm{m^3 s^{-2} kg^{-1}}$, calculate the mass of the sun.

2. Use a graphing calculator or computer program to plot the curve defined by Eq. (21). (Pick representative values of the parameters a and e.) This is an example of a *polar plot*, in which the curve in a plane is defined by giving the radial distance r as a function of the angular position θ. Be sure to set the aspect ratio (= ratio of length scales on the horizontal and vertical axes) equal to 1.

3. Consider a particle M that moves on the xy plane. The polar coordinates (r, θ) and unit vectors ($\widehat{\mathbf{r}}$, $\widehat{\boldsymbol{\theta}}$) are defined in Fig. 4.

(a) Show that $x = r\cos\theta$ and $y = r\sin\theta$.

(b) Show that for a small displacement of M,

$$\Delta\widehat{\mathbf{r}} \approx \widehat{\boldsymbol{\theta}}\,\Delta\theta \qquad \text{and} \qquad \Delta\widehat{\boldsymbol{\theta}} \approx -\widehat{\mathbf{r}}\,\Delta\theta.$$

(c) The position vector of M is $\mathbf{x} = r\widehat{\mathbf{r}}$, which has magnitude r and direction $\widehat{\mathbf{r}}$. In general, both r and $\widehat{\mathbf{r}}$ vary with time t, as the object moves. Show that the velocity vector is

$$\mathbf{v} = \frac{dr}{dt}\widehat{\mathbf{r}} + r\frac{d\theta}{dt}\widehat{\boldsymbol{\theta}}.$$

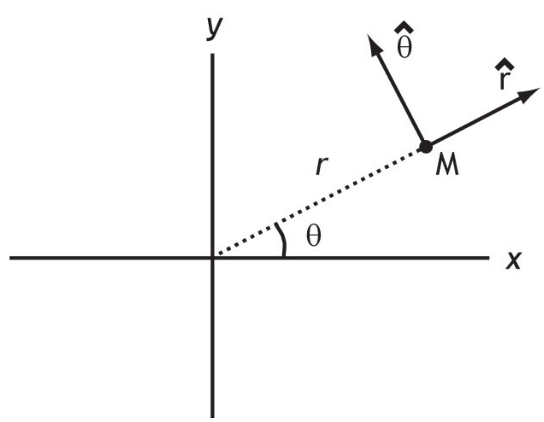


Figure 4: Problem 3.

4. Derive Eq. (22) for the radial component a_r of the acceleration in polar coordinates. [Hint: Use the results of the previous exercise.]

5. Prove that the relation of parameters in (28) is true for a circular orbit. (For a circle, the eccentricity e is 0.)

6. Look up the orbital data—period T and semimajor axis a—for the planets. Use the year (y) as the unit of time for T, and the astronomical unit (AU) as the unit of distance for a. Calculate T^2/a^3 for all nine planets. What do you notice about the values of T^2/a^3? Explain.

7. The angular momentum vector $\mathbf{L}$ for motion of a particle in three dimensions is defined by $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. Prove that $d\mathbf{L}/dt$ is equal to the torque on the particle around the origin,.

8. Reduced mass. Suppose two masses, m_1 and m_2, exert equal but opposite forces on each other. Define the center of mass position $\mathbf{R}$ and relative vector $\mathbf{r}$ by

$$\mathbf{R} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2} \qquad \text{and} \qquad \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2.$$

(Note that $\mathbf{r}$ is the vector from m_2 to m_1.)

(a) Show that $d^2\mathbf{R}/dt^2 = 0$, i.e., the center of mass point moves with constant velocity. (It could be at rest.)

(b) Show that

$$\mu \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}(\mathbf{r})$$

where μ is the *reduced mass*, $m_1 m_2/(m_1 + m_2)$. Thus the two-body problem reduces to an equivalent one-body problem with the reduced mass.

(c) Show that Kepler's third law for the case of a circular orbit should properly be

$$T^2 = \frac{4\pi^2 r^3}{G(M + m)}$$

rather than (6). Why is (6) approximately correct?

9. Consider a *binary star*. Assume the two stars move on circular orbits. Given the masses M_1 and M_2, and the distance a between the stars, determine the period of revolution T.

10. Flight to Mars. To send a satellite from Earth to Mars, a rocket must accelerate the satellite until it is in the correct elliptical orbit around the sun. The satellite does not travel to Mars under rocket power, because that would require more fuel than it could carry. It just moves on a Keplerian orbit under the influence of the sun's gravity.

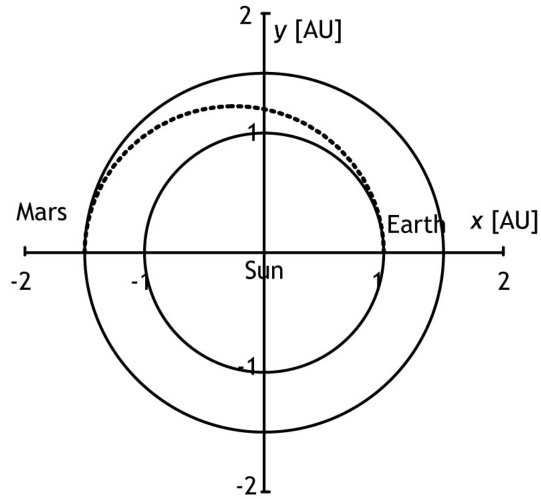
The satellite orbit must have perihelion distance $r_- = R_E$ (= radius of Earth's orbit) and aphelion distance $r_+ = R_M$ (= radius of Mars's orbit) as shown in the figure. The planetary orbit radii are

$$R_E = 1.496 \times 10^{11}\,\text{m} \qquad \text{and} \qquad R_M = 2.280 \times 10^{11}\,\text{m}.$$

(a) What is the semimajor axis of the satellite's orbit?

(b) Calculate the time for the satellite's journey. Express the result in months and days, counting one month as 30 days.

Figure 5: Problem 10.



11. Parametric plots in Mathematica

A parametric plot is a kind of graph—a curve of y versus x where x and y are known as functions of an independent variable t called *the parameter*. To plot the curve specified by

$$x = f(t) \text{ and } y = g(t),$$

the Mathematica command is

```
ParametricPlot[{f[t],g[t]},{t,t1,t2},
  PlotRange->{{x1,x2},{y1,y2}},
  AspectRatio->r]
```

Here {`t1,t2`} is the domain of t, and {`x1,x2`} and {`y1,y2`} are the ranges of x and y. To give the x and y axes equal scales, `r` should have the numerical value of `(y2-y1)/(x2-x1)`.

Use Mathematica to make the parametric plots below. In each case name the curve that results.

(a) $x(t) = t, \qquad y(t) = t - t^2.$

(b) $x(t) = t, \qquad y(t) = 1/t.$

(c) $x(t) = \cos(2\pi t), \qquad y(t) = \sin(2\pi t).$

(d) $x(t) = 2\cos(2\pi t), \qquad y(t) = 0.5\sin(2\pi t).$

(e) $x(t) = \cos(2\pi t/3), \qquad y(t) = \sin(2\pi t/7).$

12. Parametric equations for a planetary orbit

The sun is at the origin and the plane of the orbit has coordinates x and y. We can write parametric equations for the time t, and coordinates x and y, in terms of an independent variable ψ:

$$\begin{aligned} t &= \frac{T}{2\pi}(\psi - \varepsilon \sin\psi) \\ x &= a(\cos\psi - \varepsilon) \\ y &= a\sqrt{1-\varepsilon^2}\sin\psi \end{aligned}$$

The fixed parameters are T = period of revolution, a = semimajor axis, and ε = eccentricity.

(a) The orbit parameters of Halley's comet are

$$a = 17.959\,\mathrm{AU} \ \text{ and } \ \varepsilon = 0.9673.$$

Use Mathematica to make a parametric plot of the orbit of Halley's comet. (You only need the parametric equations for x and y, letting the variable ψ go from 0 to 2π for one revolution.)

(b) Calculate the perihelion distance. Express the result in AU.

(c) Calculate the aphelion distance. Express the result in AU. How does this compare to the radius of the orbit of Saturn, or Neptune?

(d) Calculate the period of revolution. Express the result in years.

13. Parametric surfaces

A parametric *curve* is a curve on a plane. The curve is specified by giving coordinates x and y as functions of an independent parameter t.

A parametric *surface* is a surface in 3 dimensions. The surface is specified by giving coordinates x, y, and z as functions of 2 independent parameters u and v. That is, the parametric equations for a surface have the form

$$x = f(u,v), \qquad y = g(u,v), \qquad z = h(u,v).$$

As u and v vary over their domains, the points (x, y, z) cover the surface.

The Mathematica command for plotting a parametric surface is `ParametricPlot3D`. To make a graph of the surface, execute the command

```
ParametricPlot3D[{f[u,v],g[u,v],h[u,v]},
   {u,u1,u2},{v,v1,v2}]
```

In this command, (u_1, u_2) is the domain of u and (v_1, v_2) is the domain of v. Before giving the command you must define in Mathematica the functions `f[u,v]`, `g[u,v]`, `h[u,v]`. For example, for exercise (a) below you would define

```
f[u_,v_]:=Sin[u]Cos[v]
```

Make plots of the following parametric surfaces. In each case name the surface.

(a) For $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$,

$$\begin{aligned} f(u,v) &= \sin u \cos v \\ g(u,v) &= \sin u \sin v \\ h(u,v) &= \cos u \end{aligned}$$

(b) For $0 \leq u \leq 2\pi$ and $-0.3 \leq v \leq 0.3$,

$$\begin{aligned} f(u,v) &= \cos u + v\cos(u/2)\cos u \\ g(u,v) &= \sin u + v\cos(u/2)\sin u \\ h(u,v) &= v\sin(u/2) \end{aligned}$$

(c) For $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$,

$$\begin{aligned} f(u,v) &= 0.2(1 - v/(2\pi))\cos(2v)(1+\cos u) + 0.1\cos(2v) \\ g(u,v) &= 0.2(1 - v/(2\pi))\sin(2v)(1+\cos u) + 0.1\sin(2v) \\ h(u,v) &= 0.2(1 - v/(2\pi))\sin u + v/(2\pi) \end{aligned}$$