

Chapter 8

continued

Rotational Dynamics

8.6 *The Action of Forces and Torques on Rigid Objects*

Chapter 8 developed the concepts of angular motion.

θ : angles and radian measure for angular variables

ω : angular velocity of rotation (same for entire object)

α : angular acceleration (same for entire object)

$v_T = \omega r$: tangential velocity

$a_T = \alpha r$: tangential acceleration

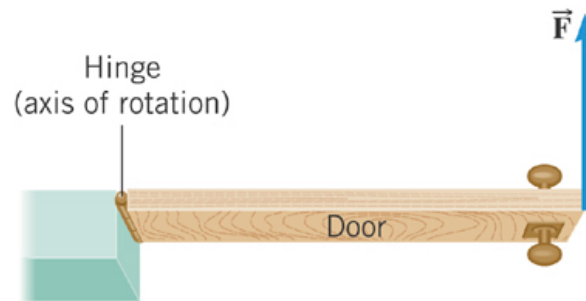
According to Newton's second law, a net force causes an object to have a ***linear acceleration***.

What causes an object to have an ***angular acceleration***?

TORQUE

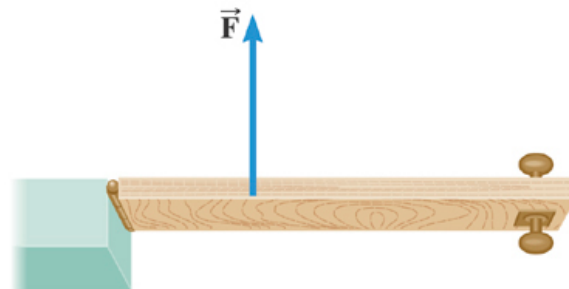
8.6 *The Action of Forces and Torques on Rigid Objects*

The amount of torque depends on where and in what direction the force is applied, as well as the location of the axis of rotation.



Maximum rotational effect of the force F .

(a)



Smaller rotational effect of the force F .

(b)



Rotational effect of the force F is minimal; it compresses more than rotates the bar

(c)

8.6 The Action of Forces and Torques on Rigid Objects

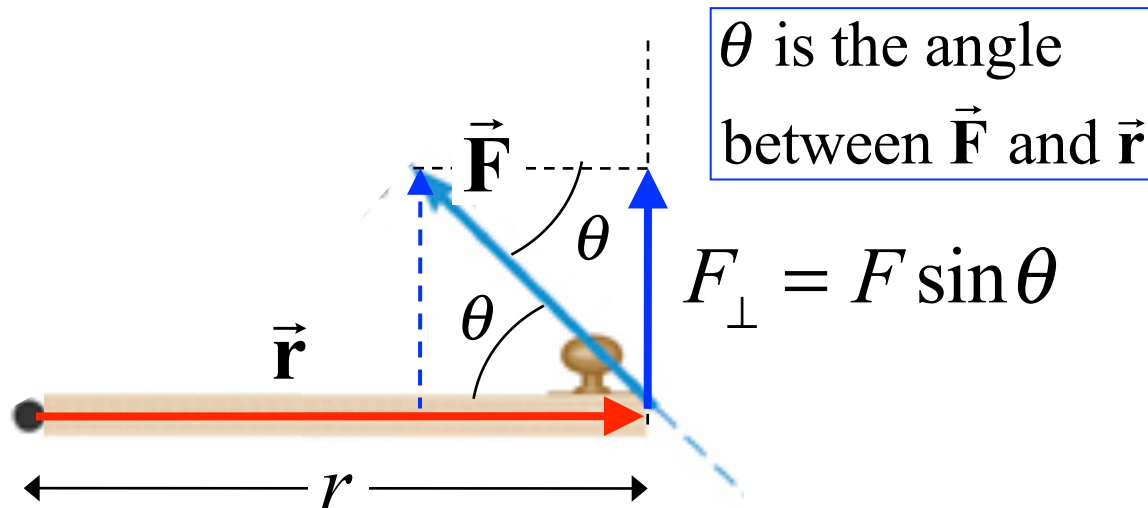
DEFINITION OF TORQUE

Magnitude of Torque = $r \times$ (Component of Force $\perp$ to $\vec{r}$)

$$\tau = rF_{\perp} = rF \sin \theta$$

Direction: The torque is **positive** when the force tends to **produce a counterclockwise rotation** about the axis.

SI Unit of Torque: newton x meter (N·m)



8.6 The Action of Forces and Torques on Rigid Objects

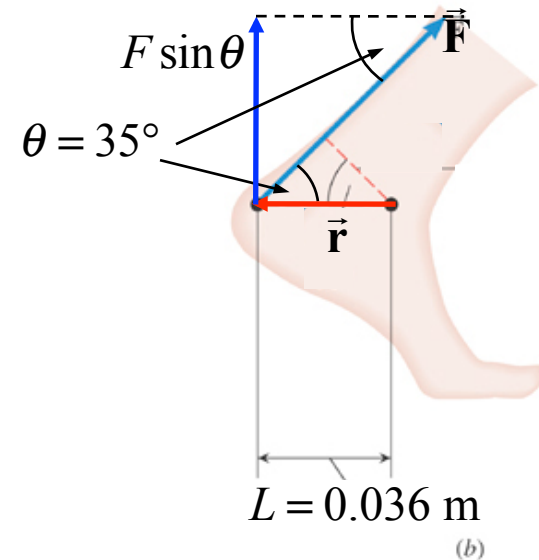
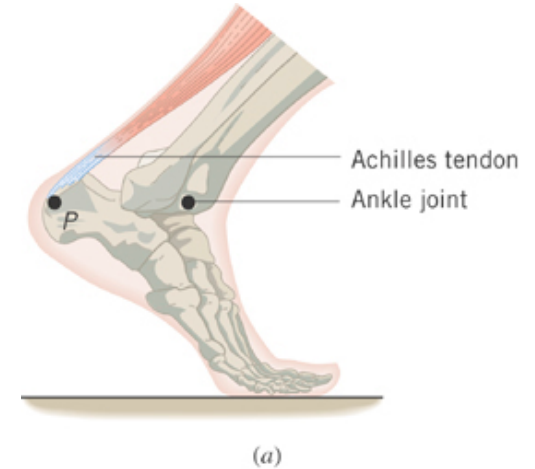
Example: The Achilles Tendon

The tendon exerts a force of magnitude 720 N. Determine the torque (magnitude and direction) of this force about the ankle joint. Assume the angle is 35° .

$$\tau = r(F \sin \theta) = (.036 \text{ m})(720 \text{ N})(\sin 35^\circ) \\ = 15.0 \text{ N} \cdot \text{m}$$

θ is the angle
between $\vec{F}$ and $\vec{r}$

Direction is clockwise (–) around ankle joint
Torque vector $\tau = -15.0 \text{ N} \cdot \text{m}$



8.7 Rigid Objects in Equilibrium

If a rigid body is in equilibrium, neither its linear motion nor its rotational motion changes.

$$a_x = a_y = 0$$

$$\alpha = 0$$

$$\sum F_x = 0 \quad \sum F_y = 0$$

$$\sum \tau = 0$$

All **equilibrium** problems use these equations – no net force and no net torque.

8.7 Rigid Objects in Equilibrium

EQUILIBRIUM OF A RIGID BODY

A rigid body is in equilibrium if it has **zero translational acceleration** and **zero angular acceleration**. In equilibrium, the sum of the externally applied forces is zero, and the sum of the externally applied torques is zero.

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum \tau = 0$$

Note: **constant linear speed** or **constant rotational speed** are allowed for an object in equilibrium.

8.7 *Rigid Objects in Equilibrium*

Reasoning Strategy

1. Select the object to which the equations for equilibrium are to be applied.
2. Draw a free-body diagram that shows all of the external forces acting on the object.
3. Choose a convenient set of x , y axes and resolve all forces into components that lie along these axes.
4. Apply the equations that specify the balance of forces at equilibrium. (Set the net force in the x and y directions equal to zero.)
5. Select a convenient axis of rotation. Set the sum of the torques about this axis equal to zero.
6. Solve the equations for the desired unknown quantities.

8.7 Rigid Objects in Equilibrium

Example A Diving Board

A woman whose weight is 530 N is poised at the right end of a diving board with length 3.90 m. The board has negligible weight and is supported by a fulcrum 1.40 m away from the left end. Find the forces that the bolt and the fulcrum exert on the board.

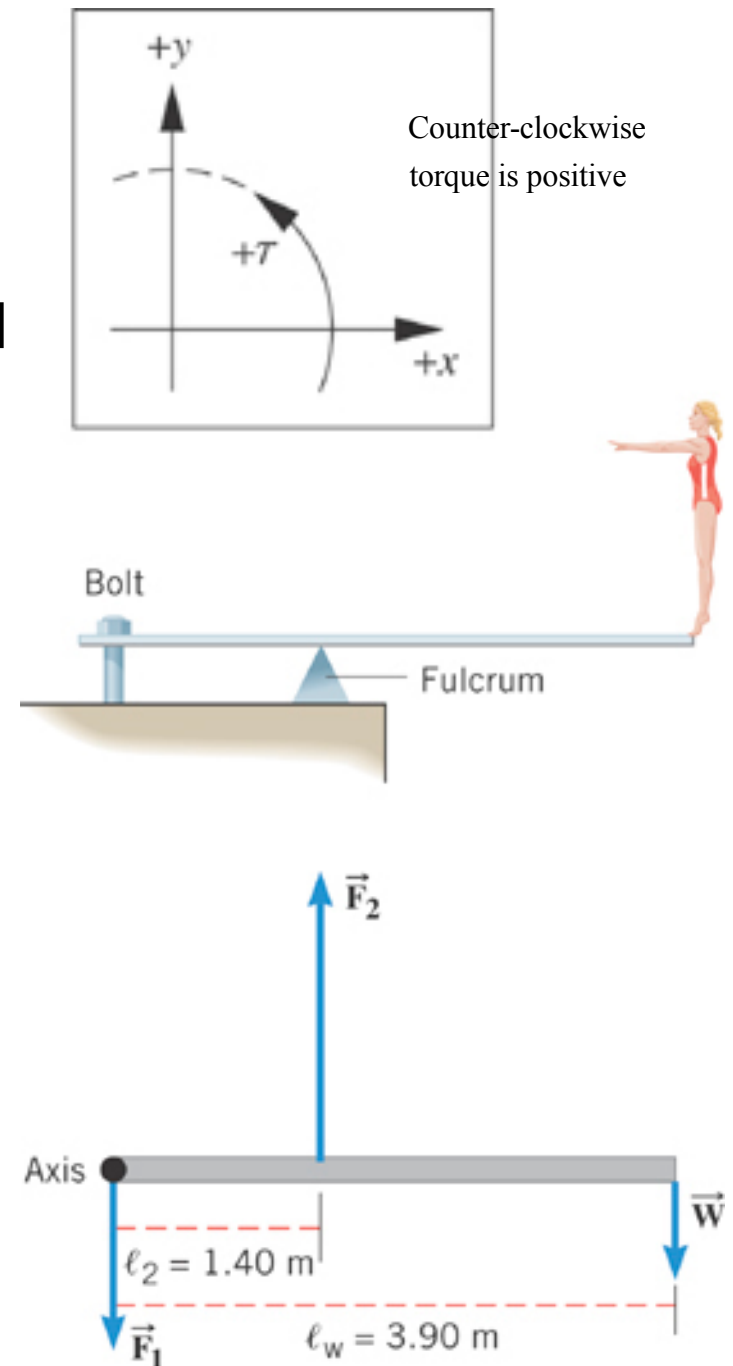
F_1 acts on rotation axis - produces no torque.

$$\sum \tau = 0 = \ell_2 F_2 - \ell_w W$$

$$F_2 = (\ell_w / \ell_2) W = (3.9/1.4) 530 \text{ N} = 1480 \text{ N}$$

$$\sum F_y = 0 = -F_1 + F_2 - W$$

$$F_1 = F_2 - W = (1480 - 530) \text{ N} = 950 \text{ N}$$



8.7 Rigid Objects in Equilibrium

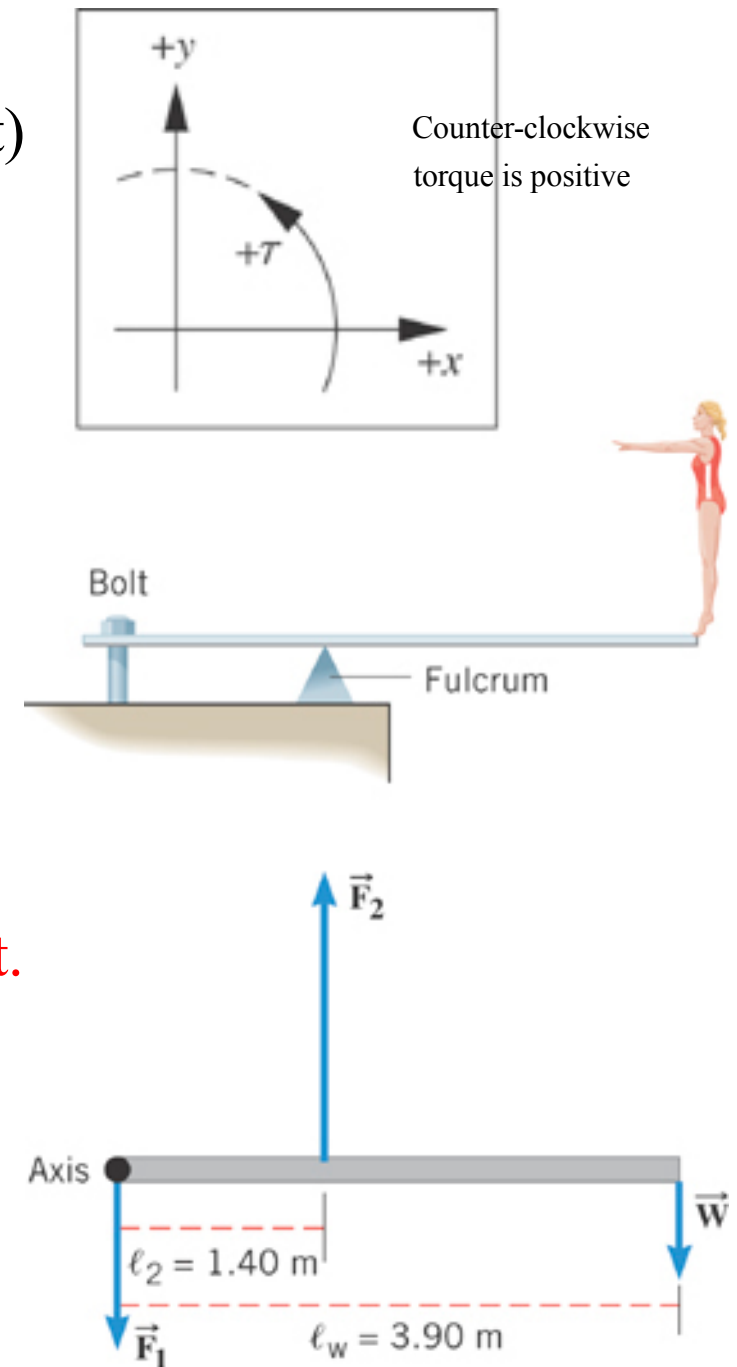
Choice of pivot is arbitrary (most convenient)

Pivot at fulcrum: F_2 produces no torque.

$$\sum \tau = 0 = F_1 \ell_2 - W(\ell_w - \ell_2)$$
$$F_1 = W(\ell_w / \ell_2 - 1) = (530\text{N})(1.8) = 950\text{ N}$$

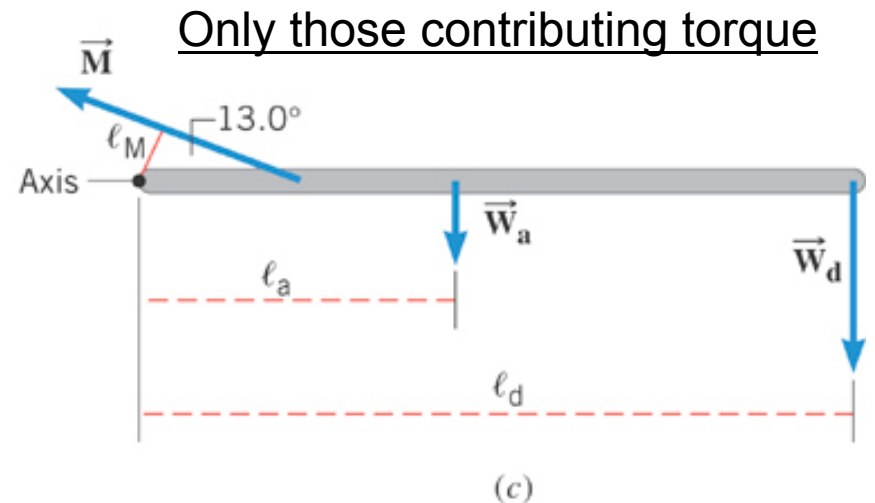
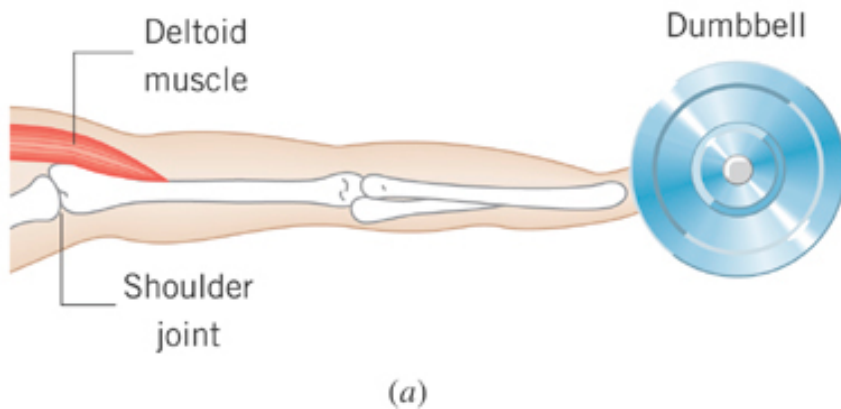
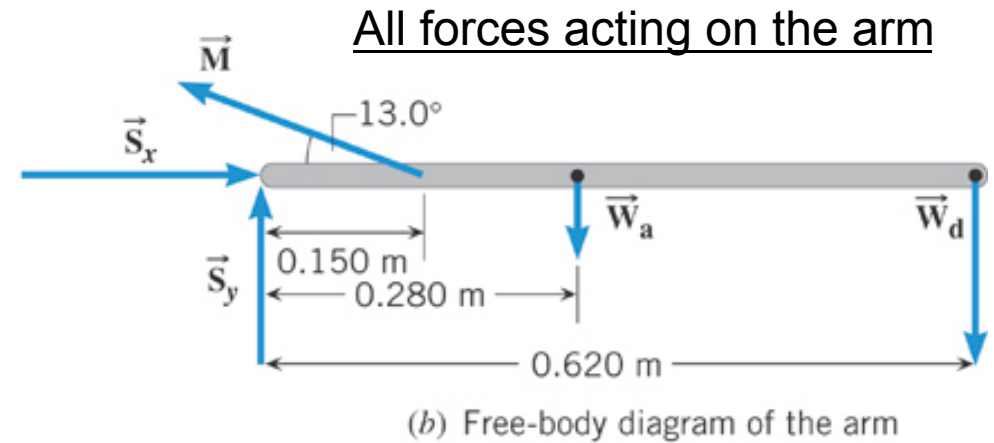
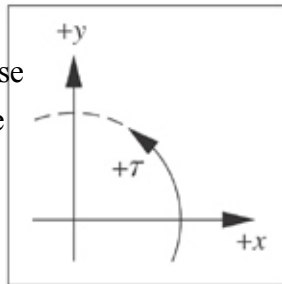
$$\sum F_y = 0 = -F_1 + F_2 - W$$
$$F_2 = F_1 + W = (950 + 530)\text{N} = 1480\text{ N}$$

Yields the same answers as with pivot at Bolt.



8.7 Rigid Objects in Equilibrium

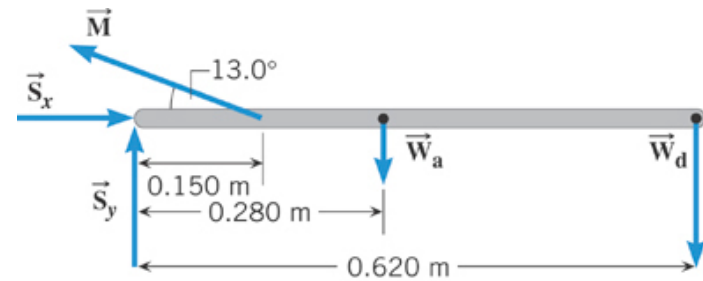
Counter-clockwise torque is positive



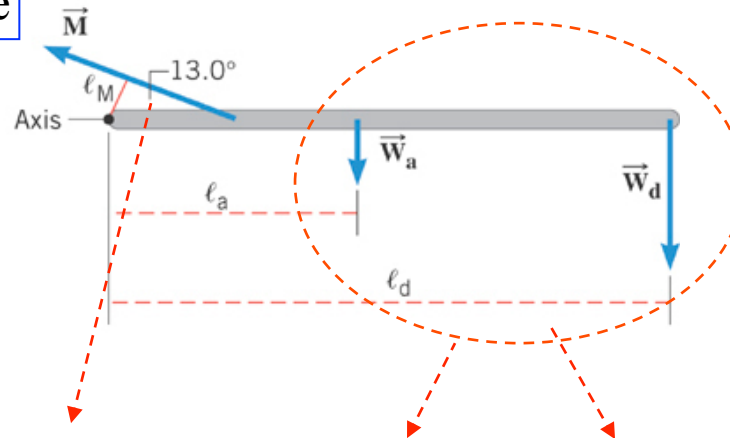
Example 5 Bodybuilding

The arm is horizontal and weighs 31.0 N. The deltoid muscle can supply 1840 N of force. What is the weight of the heaviest dumbbell he can hold?

8.7 Rigid Objects in Equilibrium



positive torque

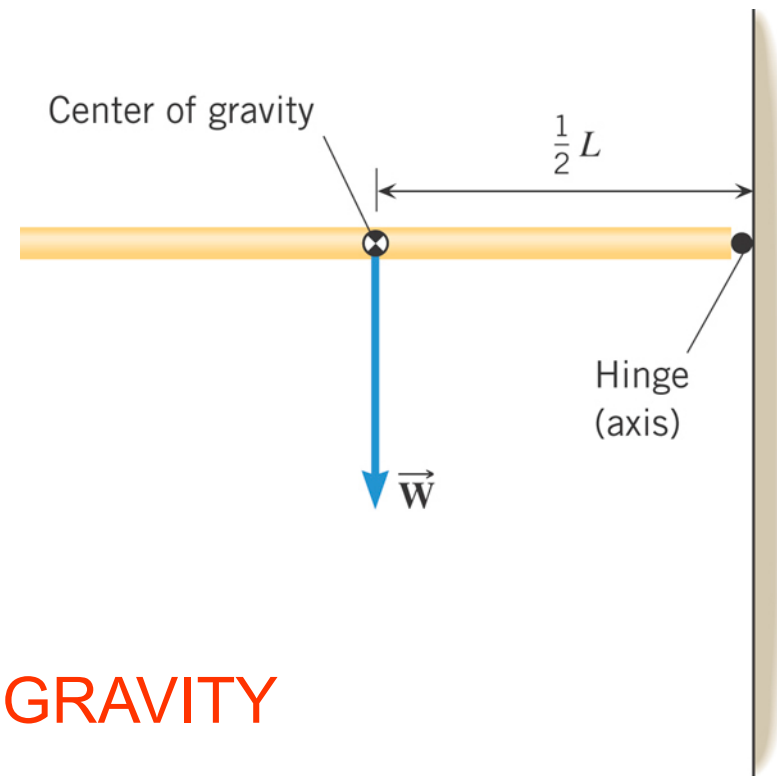


negative torques

$$\sum \tau = M(\sin 13^\circ)\ell_M - W_a\ell_a - W_d\ell_d = 0$$

$$\begin{aligned} W_d &= \left[+M(\sin 13^\circ)\ell_M - W_a\ell_a \right] / \ell_d \\ &= \left[1840\text{N}(.225)(0.15\text{m}) - 31\text{N}(0.28\text{m}) \right] / 0.62\text{m} \\ &= 86.1\text{N} \end{aligned}$$

8.7 Center of Gravity

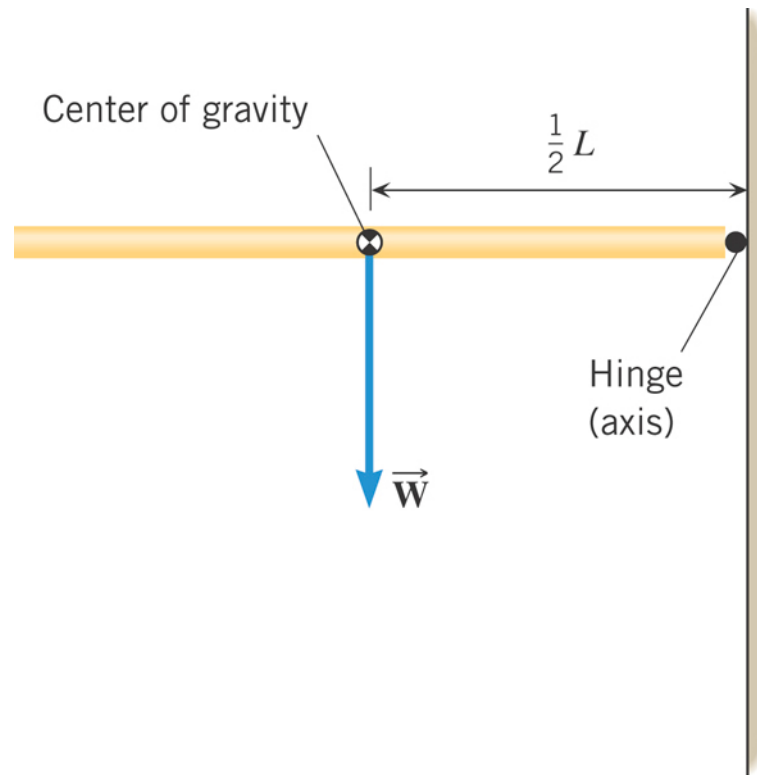


DEFINITION OF CENTER OF GRAVITY

The center of gravity of a rigid body is the point at which its weight can be considered to act when the torque due to the weight is being calculated.

8.7 Center of Gravity

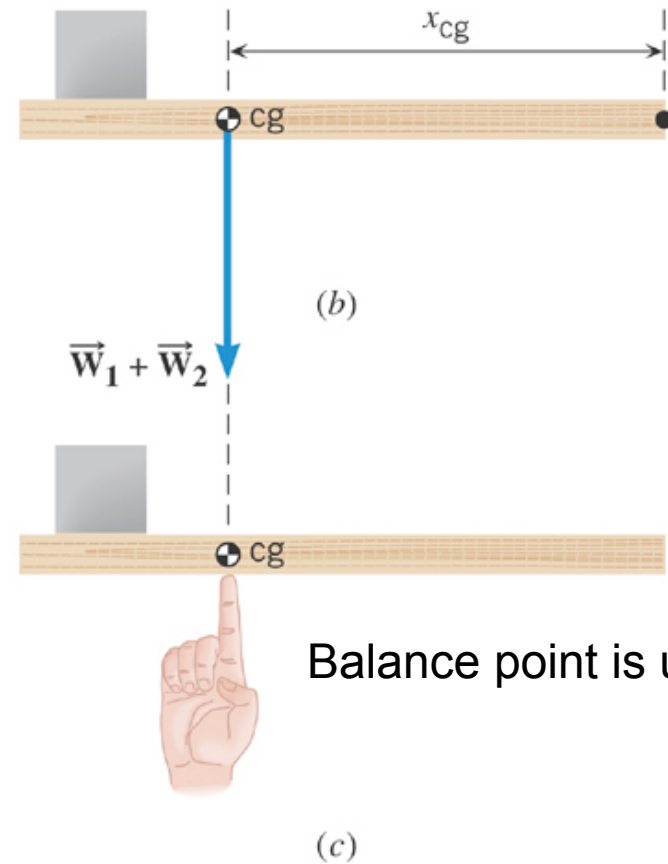
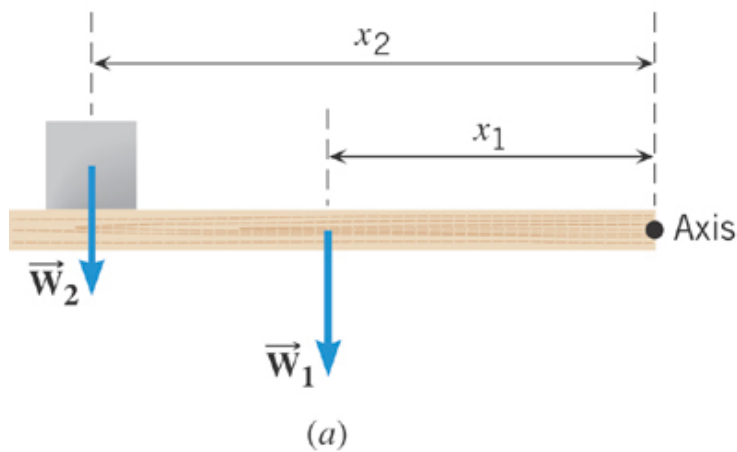
When an object has a symmetrical shape and its weight is distributed uniformly, the center of gravity lies at its geometrical center.



8.7 Center of Gravity

General Form of x_{cg}

$$x_{cg} = \frac{W_1 x_1 + W_2 x_2 + \dots}{W_1 + W_2 + \dots}$$



Center of Gravity, x_{cg} , for 2 masses

$$x_{cg} = \frac{W_1 x_1 + W_2 x_2}{W_1 + W_2}$$

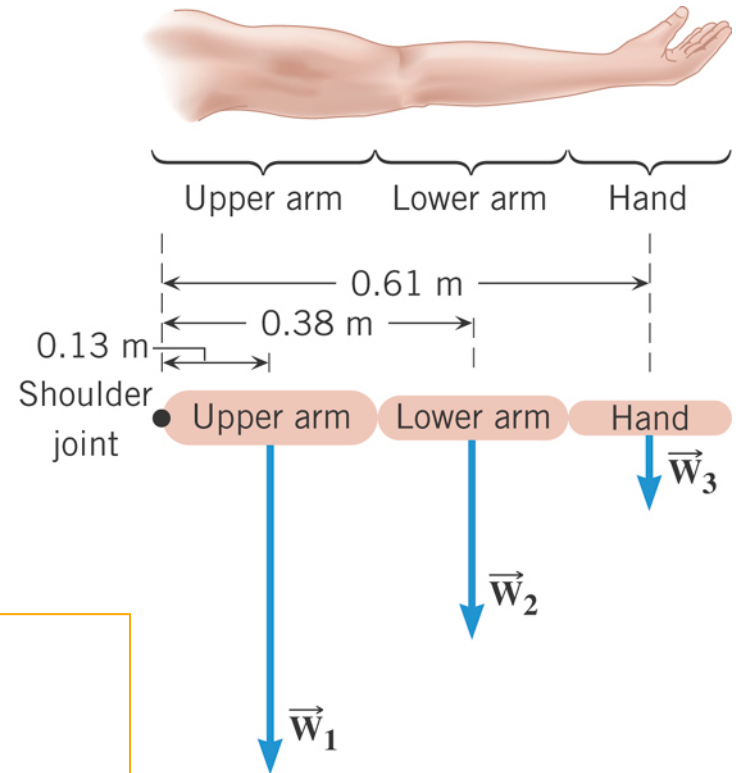
8.7 Center of Gravity

Example: The Center of Gravity of an Arm

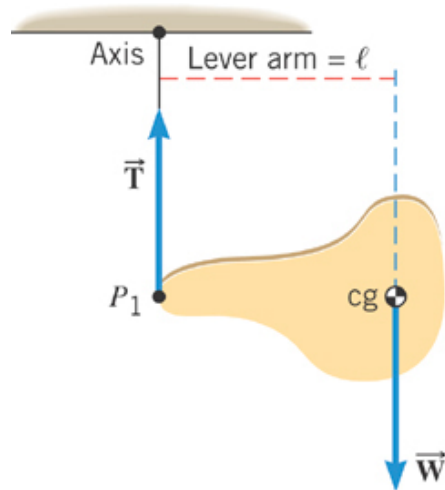
The horizontal arm is composed of three parts: the upper arm (17 N), the lower arm (11 N), and the hand (4.2 N).

Find the center of gravity of the arm relative to the shoulder joint.

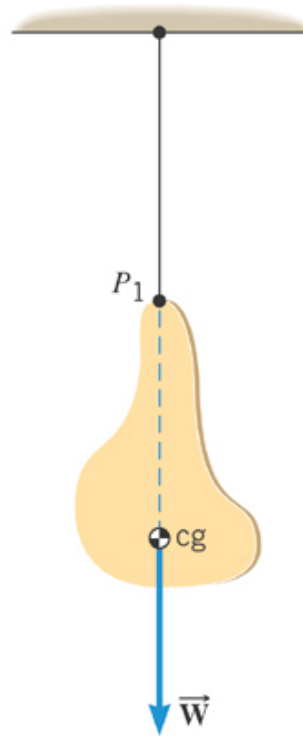
$$\begin{aligned}x_{cg} &= \frac{W_1x_1 + W_2x_2 + W_3x_3}{W_1 + W_2 + W_3} \\&= \frac{[17(0.13) + 11(0.38) + 4.2(0.61)] \text{ N} \cdot \text{m}}{(17 + 11 + 4.2) \text{ N}} = 0.28 \text{ m}\end{aligned}$$



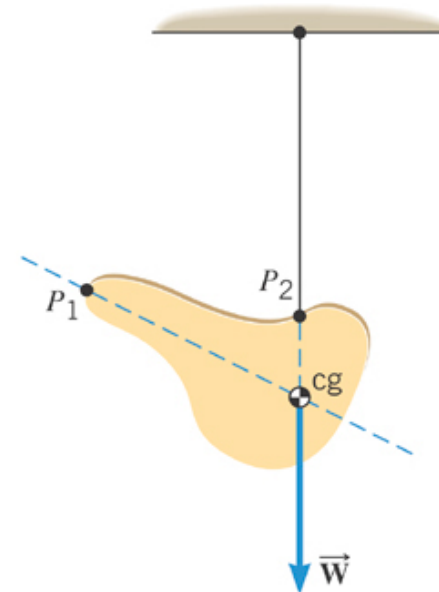
8.7 Center of Gravity



(a)



(b)



(c)

Finding the center of gravity of an irregular shape.

Clicker Question 8.4 Torque and Equilibrium (no clickers today)

A 4-kg ball and a 1-kg ball are positioned a distance L apart on a bar of negligible mass. How far from the 4-kg mass should the fulcrum be placed to balance the bar?

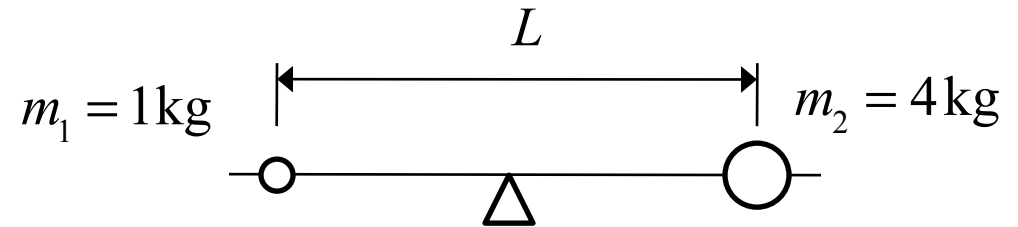
a) $\frac{1}{2} L$

b) $\frac{1}{3} L$

c) $\frac{1}{4} L$

d) $\frac{1}{5} L$

e) $\frac{1}{6} L$



Clicker Question 8.4 Torque and Equilibrium

A 4-kg ball and a 1-kg ball are positioned a distance L apart on a bar of negligible mass. How far from the 4-kg mass should the fulcrum be placed to balance the bar?

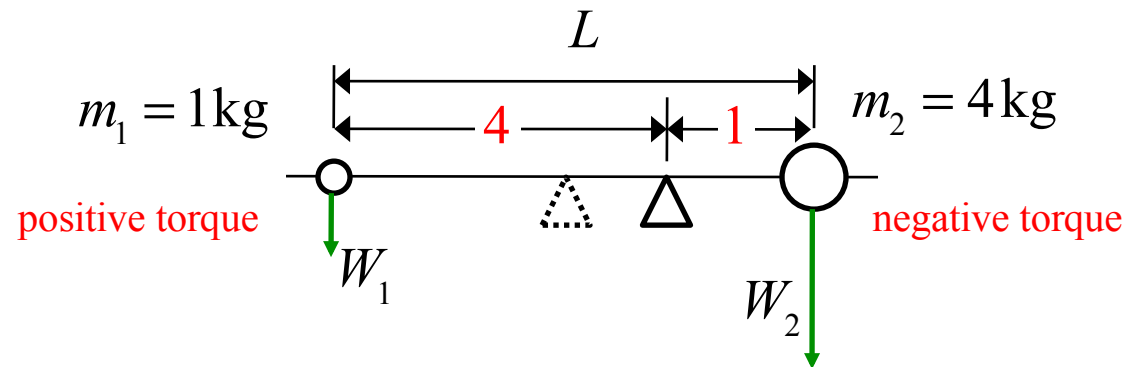
a) $\frac{1}{2} L$

b) $\frac{1}{3} L$

c) $\frac{1}{4} L$

d) $\frac{1}{5} L$

e) $\frac{1}{6} L$



For equilibrium the sum of the torques must be zero

Need to separate length into 4 parts on 1-kg mass side and 1 part on the 4-kg mass side. Total is 5 parts.

Fulcrum must be $\frac{1}{5}$ of the total length from the 4-kg mass.

Clicker Question 8.4 Torque and Equilibrium

A 4-kg ball and a 1-kg ball are positioned a distance L apart on a bar of negligible mass. How far from the 4-kg mass should the fulcrum be placed to balance the bar?

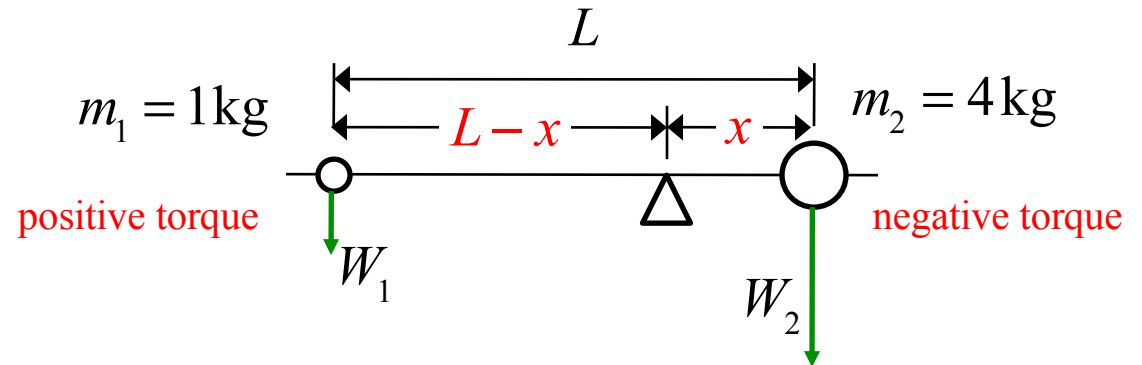
a) $\frac{1}{2} L$

b) $\frac{1}{3} L$

c) $\frac{1}{4} L$

d) $\frac{1}{5} L$

e) $\frac{1}{6} L$



For equilibrium the sum of the torques must be zero

Let x be the distance of fulcrum from 4-kg mass.

$$\sum \tau = 0 = m_1 g (L - x) + (-m_2 g x)$$

$$(m_1 + m_2) x = m_1 L$$

$$x = \frac{m_1}{(m_1 + m_2)} L = \frac{1}{(1 + 4)} L = \frac{1}{5} L$$

8.8 Newton's Second Law for Rotational Motion About a Fixed Axis

$$\tau = F_T r$$

$$= m a_T r$$

$$= m \alpha r^2$$

$$= (m r^2) \alpha$$

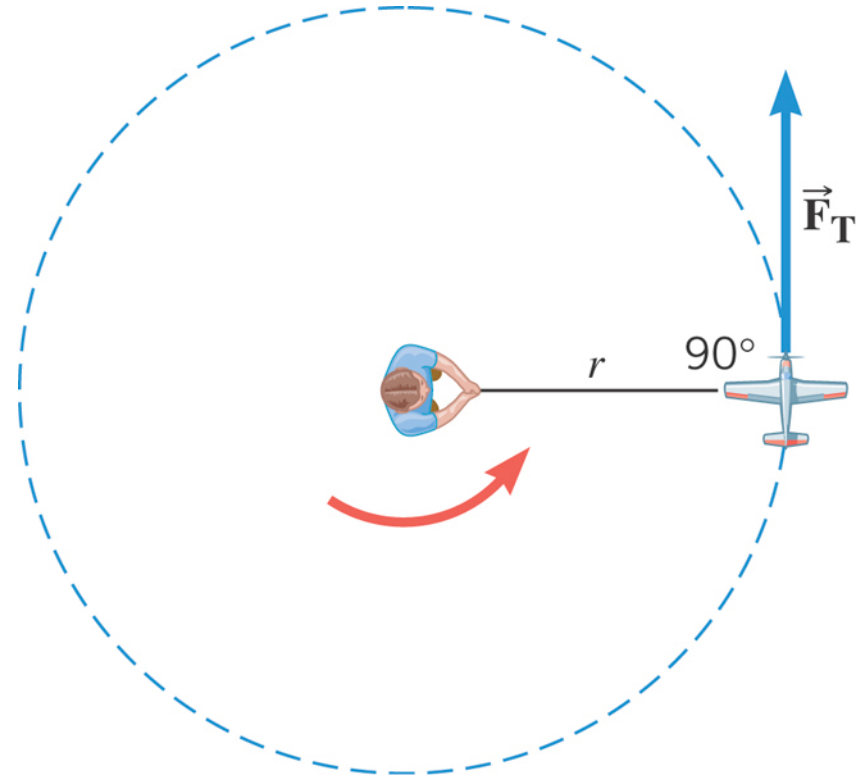
$$= I \alpha$$

$$F_T = m a_T$$

$$a_T = r \alpha$$

$$I = m r^2$$

Moment of Inertia

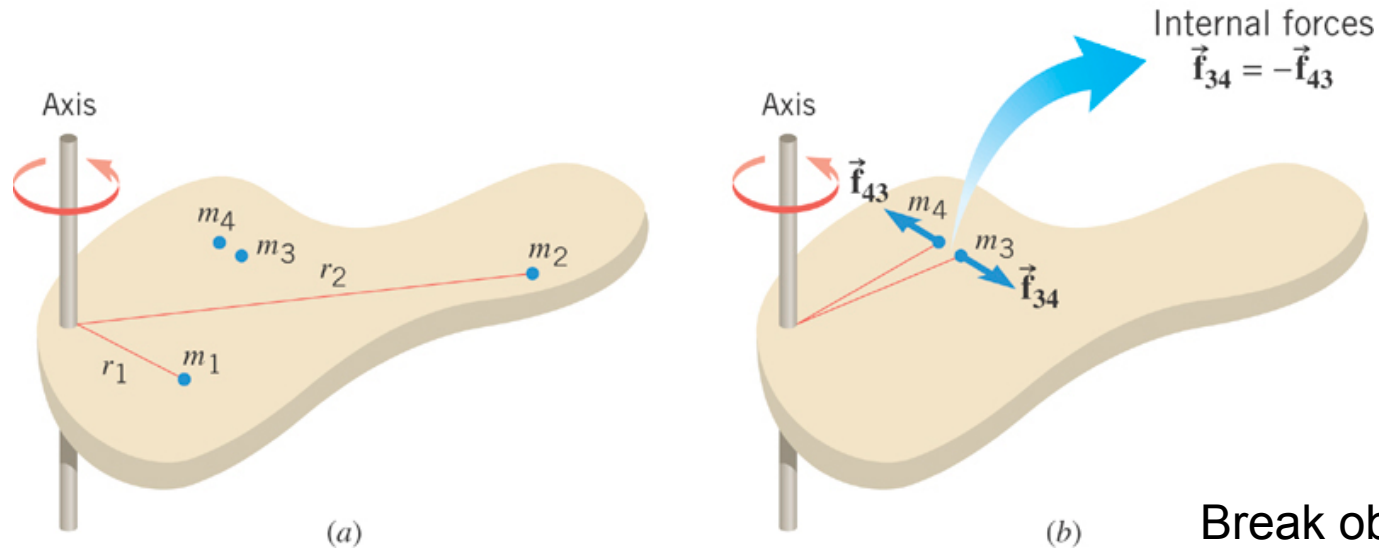


Moment of Inertia, $I = m r^2$, for a point-mass, m , at the end of a massless arm of length, r .

$$\tau = I \alpha$$

Newton's 2nd Law for rotations

8.8 Newton's Second Law for Rotational Motion About a Fixed Axis



Break object into N individual masses

$$\tau_{\text{Net}} = \sum_i (mr^2)_i \alpha$$

Net external
torque

Moment of
inertia

$$\tau_1 = (m_1 r_1^2) \alpha$$

$$\tau_2 = (m_2 r_2^2) \alpha$$

$\vdots$

$$\tau_N = (m_N r_N^2) \alpha$$

8.8 *Newton's Second Law for Rotational Motion About a Fixed Axis*

ROTATIONAL ANALOG OF NEWTON'S SECOND LAW FOR A RIGID BODY ROTATING ABOUT A FIXED AXIS

$$\text{Net external torque} = \left(\begin{array}{c} \text{Moment of} \\ \text{inertia} \end{array} \right) \times \left(\begin{array}{c} \text{Angular} \\ \text{acceleration} \end{array} \right)$$

$$\tau_{\text{Net}} = I \alpha$$

$$I = \sum_i (mr^2)_i$$

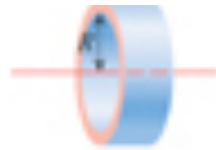
Requirement: Angular acceleration must be expressed in radians/s².

From Ch.8.4 Rotational Kinetic Energy

Moments of Inertia of Rigid Objects Mass M

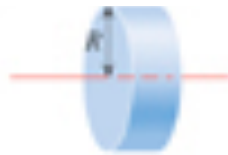
Thin walled hollow cylinder

$$I = MR^2$$



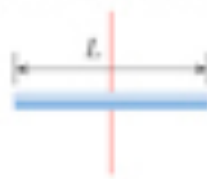
Solid cylinder or disk

$$I = \frac{1}{2} MR^2$$



Thin rod length ℓ through center

$$I = \frac{1}{12} M\ell^2$$

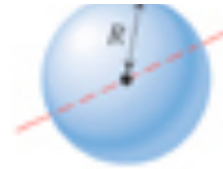


Thin rod length ℓ through end

$$I = \frac{1}{3} M\ell^2$$

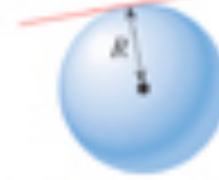


Solid sphere through center



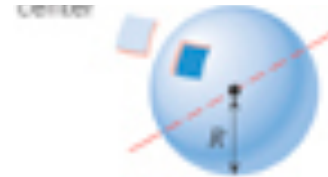
$$I = \frac{2}{5} MR^2$$

Solid sphere through surface tangent



$$I = \frac{7}{5} MR^2$$

Thin walled sphere through center



$$I = \frac{2}{3} MR^2$$

Thin plate width ℓ , through center



$$I = \frac{1}{12} M\ell^2$$

Thin plate width ℓ through edge



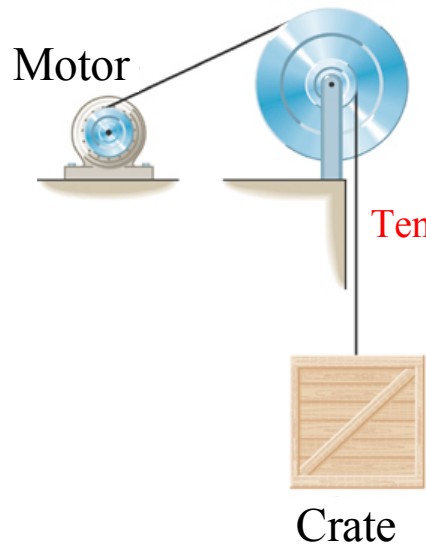
$$I = \frac{1}{3} M\ell^2$$

8.8 Newton's Second Law for Rotational Motion About a Fixed Axis

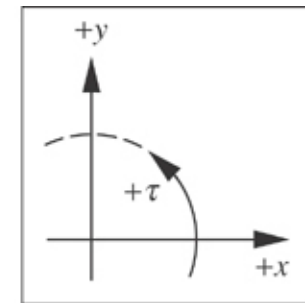
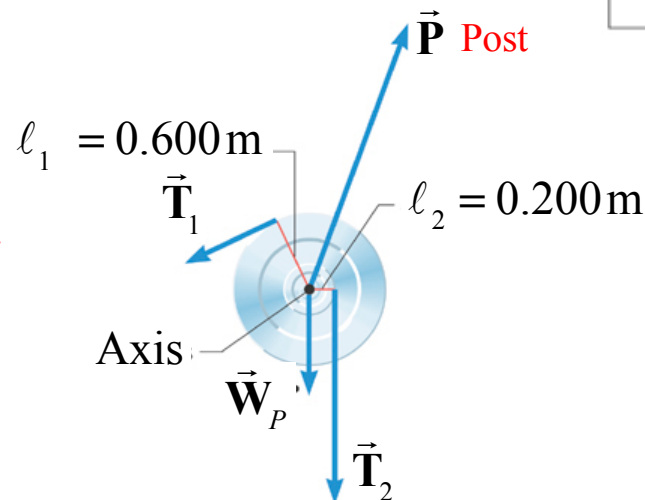
Example: Hoisting a Crate

The combined moment of inertia of the dual pulley is $50.0 \text{ kg}\cdot\text{m}^2$. The crate weighs 4420 N . A tension of 2150 N is maintained in the cable attached to the motor. Find the angular acceleration of the dual Pulley (radius-1 = 0.600 m , radius-2 = 0.200 m).

Tension 1 Dual Pulley

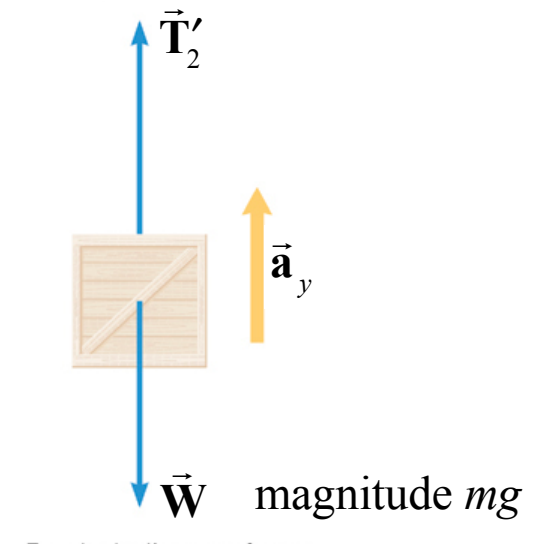


Forces on Pulley



up is positive

Forces on Crate



8.8 Newton's Second Law for Rotational Motion About a Fixed Axis

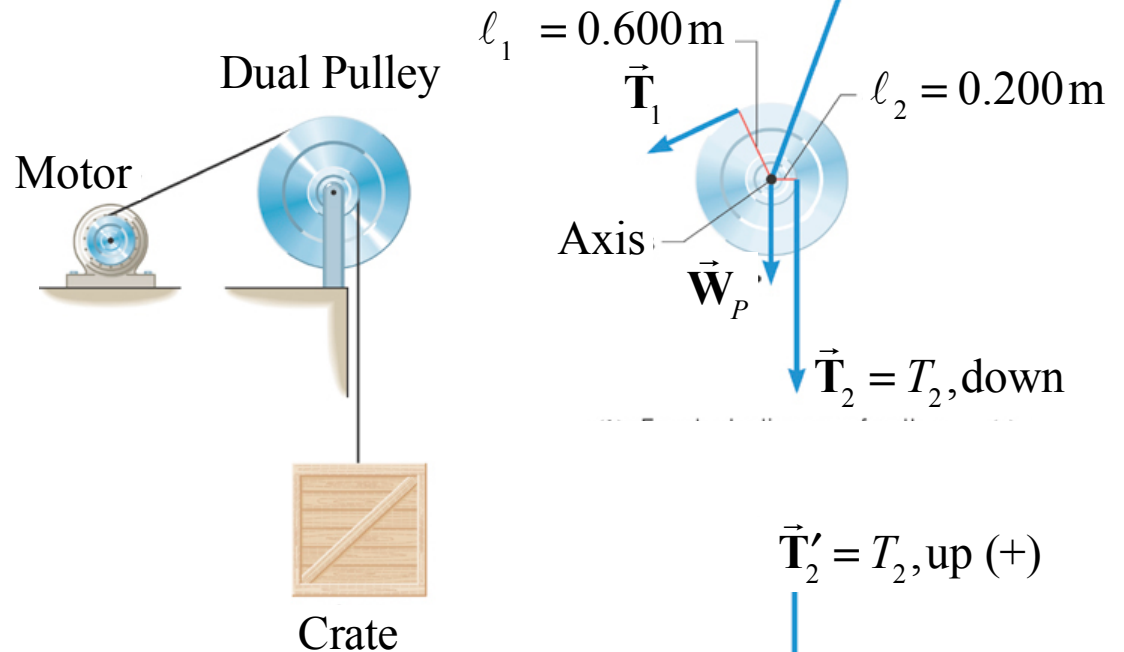
2nd law for linear motion of crate

$$\sum F_y = T_2 - mg = ma_y \quad a_y = \ell_2 \alpha$$

$$\begin{aligned} I &= 46 \text{ kg} \cdot \text{m}^2 \\ mg &= 4420 \text{ N} \\ T_1 &= 2150 \text{ N} \end{aligned}$$

2nd law for rotation of the pulley

$$\sum \tau = T_1 \ell_1 - T_2 \ell_2 = I \alpha$$



$$T_2 - mg = m\ell_2 \alpha$$

$$T_1 \ell_1 - T_2 \ell_2 = I \alpha$$

$$T_2 = m\ell_2 \alpha + mg \quad \& \quad T_2 = \frac{T_1 \ell_1 - I \alpha}{\ell_2}$$

$$m\ell_2^2 \alpha + I \alpha = T_1 \ell_1 - mg \ell_2$$

$$\alpha = \frac{T_1 \ell_1 - mg \ell_2}{m\ell_2^2 + I} = 6.3 \text{ rad/s}^2$$

$$T_2 = m\ell_2 \alpha + mg = (451(.6)(6.3) + 4420) \text{ N} = 6125 \text{ N}$$

8.9 *Angular Momentum*

DEFINITION OF ANGULAR MOMENTUM

The angular momentum L of a body rotating about a fixed axis is the product of the body's moment of inertia and its angular velocity with respect to that axis:

$$L = I\omega$$

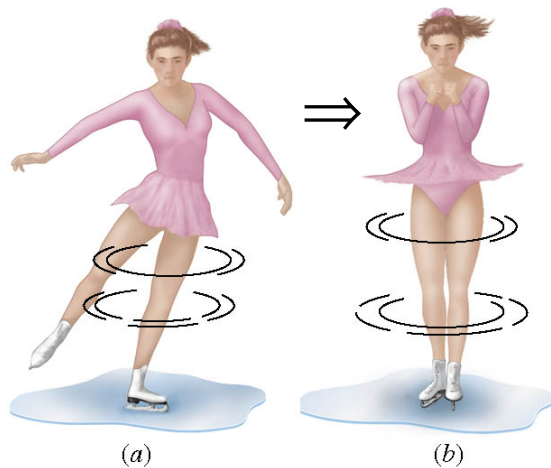
Requirement: The angular speed must be expressed in rad/s.

SI Unit of Angular Momentum: $\text{kg}\cdot\text{m}^2/\text{s}$

8.9 Angular Momentum

PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM

The angular momentum of a system remains constant (is conserved) if the **net external torque acting on the system is zero**.



Moment of Inertia
decreases

$$I = \sum mr^2, \quad r_f < r_i$$
$$I_f < I_i$$
$$\frac{I_i}{I_f} > 1$$

Angular momentum, L

$$L_i = I_i \omega_i; \quad L_f = I_f \omega_f$$

No external torque

$\Rightarrow$ Angular momentum conserved

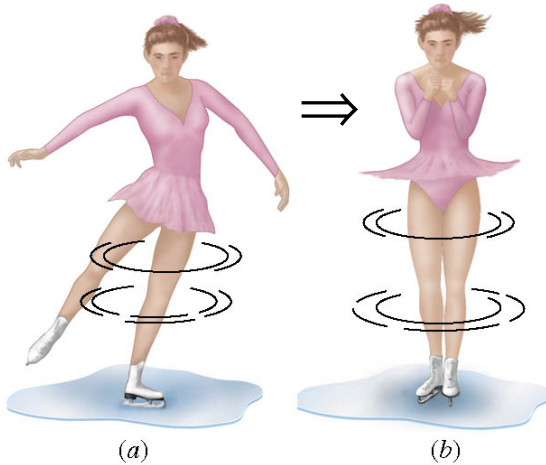
$$L_f = L_i$$

$$I_f \omega_f = I_i \omega_i$$

$$\omega_f = \frac{I_i}{I_f} \omega_i; \quad \frac{I_i}{I_f} > 1$$

$\omega_f > \omega_i$ (angular speed increases)

8.9 Angular Momentum



From Angular Momentum Conservation

$$\omega_f = \left(I_i / I_f \right) \omega_i \quad I_i / I_f > 1$$

Angular velocity increases

Is Energy conserved?

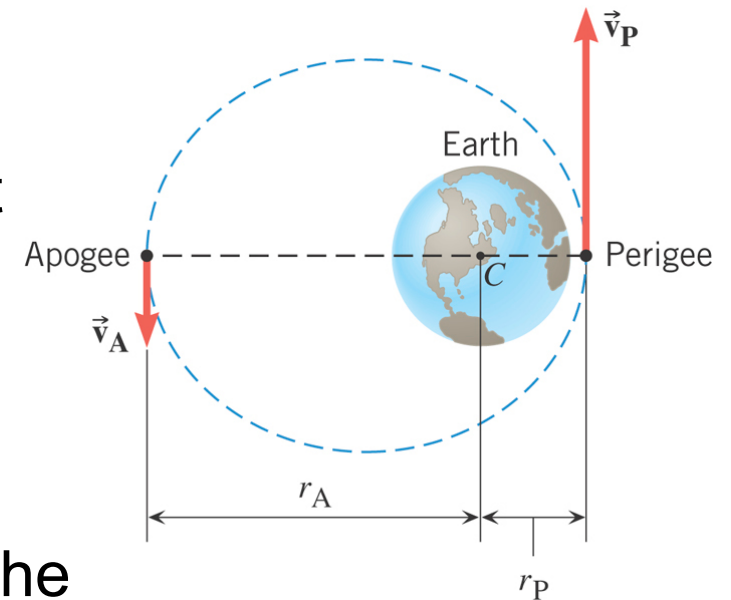
$$\begin{aligned} KE_f &= \frac{1}{2} I_f \omega_f^2 \\ &= \frac{1}{2} I_f \left(I_i / I_f \right)^2 \omega_i^2 \\ &= \left(I_i / I_f \right) \left(\frac{1}{2} I_i \omega_i^2 \right) \quad KE_i = \frac{1}{2} I_i \omega_i^2; \\ &= \left(I_i / I_f \right) KE_i \Rightarrow \text{Kinetic Energy increases} \end{aligned}$$

Energy is **not conserved** because pulling in the arms does
(NC) work on their mass and **increases the kinetic energy** of rotation

8.9 Angular Momentum

Example: A Satellite in an Elliptical Orbit

An artificial satellite is placed in an elliptical orbit about the earth. Its point of closest approach is $8.37 \times 10^6 \text{ m}$ from the center of the earth, and its point of greatest distance is $25.1 \times 10^6 \text{ m}$ from the center of the earth. The speed of the satellite at the perigee is 8450 m/s . Find the speed at the apogee.



$$I_A = mr_A^2; \quad I_P = mr_P^2$$
$$\omega_A = v_A / r_A; \quad \omega_P = v_P / r_P$$

Gravitational force along L (no torque) $\Rightarrow$ Angular momentum conserved

$$I_A \omega_A = I_P \omega_P$$

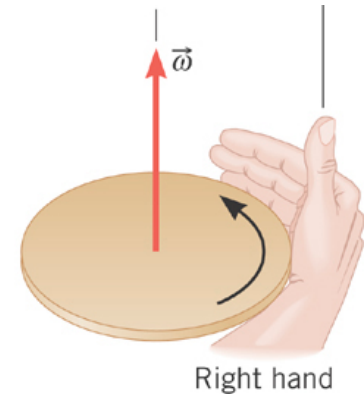
$$mr_A^2 (v_A / r_A) = mr_P^2 (v_P / r_P)$$

$$r_A v_A = r_P v_P \quad \Rightarrow \quad v_A = (r_P / r_A) v_P = \left[(8.37 \times 10^6) / (25.1 \times 10^6) \right] (8450 \text{ m/s}) = 2820 \text{ m/s}$$

8.9 The Vector Nature of Angular Variables

Right-Hand Rule: Grasp the axis of rotation with your right hand, so that your fingers circle the axis in the same sense as the rotation.

Your extended thumb points along the axis in the direction of the angular velocity.

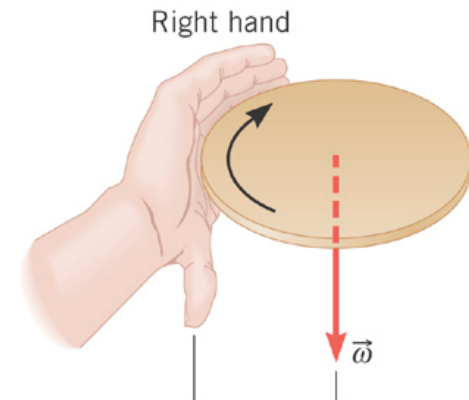


Vector Quantities in Rotational Motion

Angular Acceleration $\vec{\alpha} = \frac{\Delta \vec{\omega}}{\Delta t}$

Angular Momentum $\vec{L} = I \vec{\omega}$

Torque $\vec{\tau} = I \vec{\alpha} = \frac{\Delta \vec{L}}{\Delta t}$ Changes Angular Momentum



If torque is perpendicular to the angular momentum, only the direction of the angular momentum changes (precession) – no changes to the magnitude.

Rotational/Linear Dynamics Summary

<u>linear</u>	<u>rotational</u>	<u>linear</u>	<u>rotational</u>
x	θ	$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$	$\vec{\tau} = I\vec{\alpha}$
v	ω	$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$	$\vec{L} = I\vec{\omega}$
a	α	$W = rF \cos \theta$	$W_{rot} = \tau\theta$
m	$I = mr^2$ (point m)	$KE = \frac{1}{2}mv^2$	$KE_{rot} = \frac{1}{2}I\omega^2$
F	$\tau = Fr \sin \theta$	$W \Rightarrow \Delta KE$	$W_{rot} \Rightarrow \Delta KE_{rot}$
p	$L = I\omega$	$\vec{\mathbf{F}}\Delta t \Rightarrow \Delta \vec{\mathbf{p}}$	$\vec{\tau}\Delta t \Rightarrow \Delta \vec{L}$

Potential Energies

$$U_G = mgy$$

$$\text{or } U_G = -GM_E m/R_E$$

$$U_S = \frac{1}{2}kx^2$$

Conservation laws

Conserved
quantity:

If $W_{\text{NC}} = 0$,
 $E = K + U$

If $\mathbf{F}_{\text{ext}} = 0$,
 $\mathbf{P}_{\text{system}} = \sum \mathbf{p}$

If $\tau_{\text{ext}} = 0$,
 $\vec{L} = I\vec{\omega}$