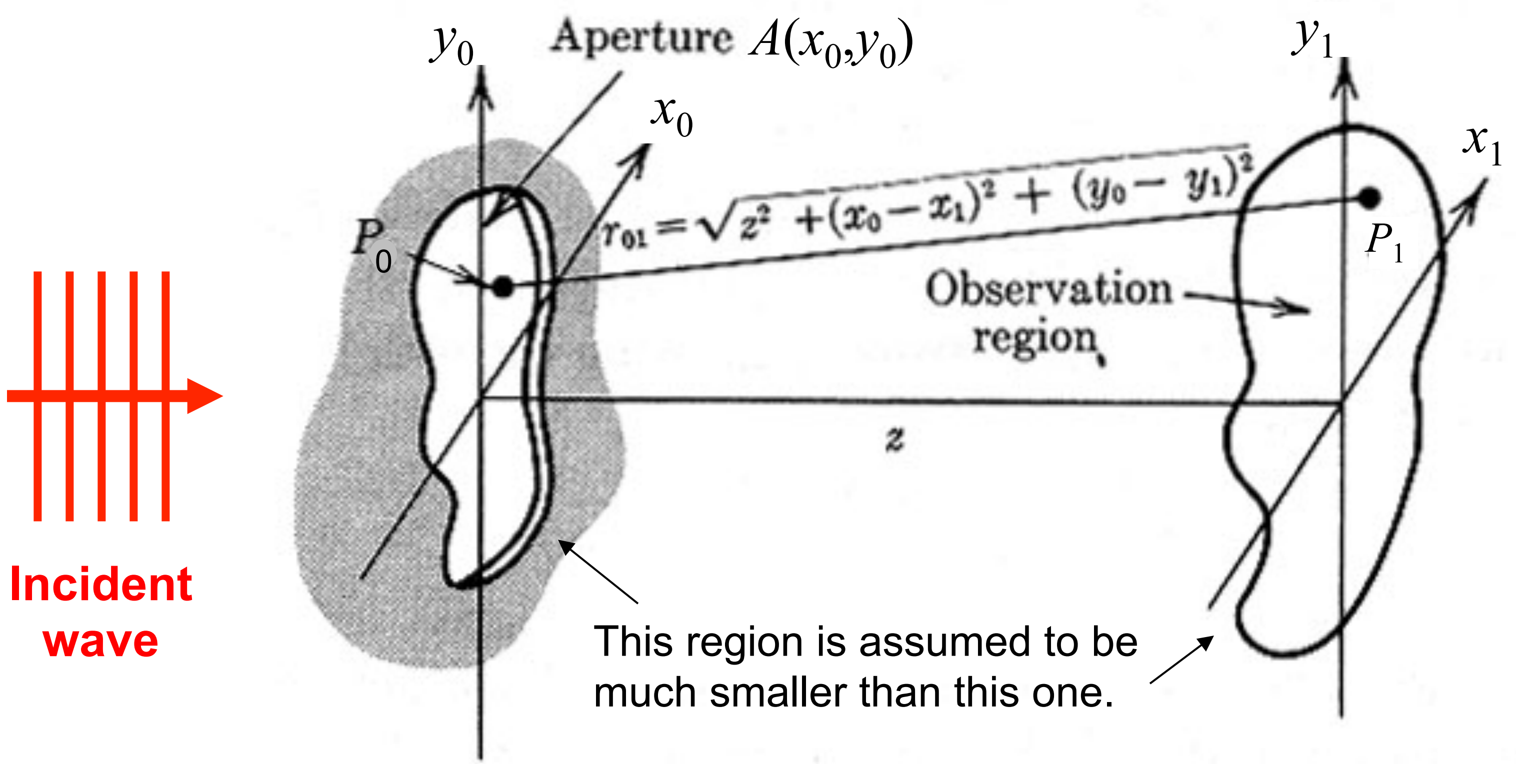
Diffraction Geometry

We wish to find the light electric field after a screen with a hole in it. This is a very general problem with far-reaching applications.



What is $E(x_1,y_1)$ at a distance z from the plane of the aperture?

Diffraction Solution

The field in the observation plane, $E(x_1,y_1)$, at a distance z from the aperture plane is given by:

$$E(x_1, y_1, z) = \iint\limits_{A(x_0, y_0)} h(x_1 - x_0, y_1 - y_0, z) E(x_0, y_0)\, dx_0\, dy_0$$

where :

$$h(x_1 - x_0, y_1 - y_0, z) = \frac{1}{i\lambda} \frac{\exp(ikr_{01})}{r_{01}}$$

Spherical wave

and :

$$r_{01} = \sqrt{z^2 + \left(x_0 - x_1\right)^2 + \left(y_0 - y_1\right)^2}$$

A very complicated result! And we cannot approximate r_{01} in the exp by z because it gets multiplied by k, which is big, so relatively small changes in r_{01} can make a big difference!

Fraunhofer Diffraction: The Far Field

We can approximate r_{01} in the denominator by z, and if D is the size of the aperture, $D^2 \geq x_0{}^2 + y_0{}^2$, so when $k D^2/ 2z << 1$, the quadratic terms << 1, so we can neglect them:

$$r_{01} = \sqrt{z^2 + \left(x_0 - x_1\right)^2 + \left(y_0 - y_1\right)^2} \approx z\left[1 + \left(x_0 - x_1\right)^2 / 2z^2 + \left(y_0 - y_1\right)^2 / 2z^2\right]$$

$$kr_{01} \approx kz + k\left(x_0^2 - 2x_0x_1 + x_1^2\right) / 2z + k\left(y_0^2 - 2y_0y_1 + y_1^2\right) / 2z$$

Small, so neglect these terms.

Independent of x_0 and y_0, so factor these out.

$$E\left(x_1, y_1\right) = \frac{\exp(ikz)}{i\lambda z}\exp\left[ik\frac{x_1^2 + y_1^2}{2z}\right]\iint_{A(x_0,y_0)} \exp\left\{-\frac{ik}{z}\left(x_0x_1 + y_0y_1\right)\right\} E\left(x_0, y_0\right) dx_0\, dy_0$$

This condition means going a distance away: $z \; >> \; kD^2 / 2 = \pi D^2 / \lambda$

If $D = 1$ mm and $\lambda = 1\ \mu m$, then $z >> 3$ m.

Fraunhofer Diffraction

We'll neglect the phase factors, and we'll explicitly write the aperture function in the integral:

$$E(x_1, y_1) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{ik}{z}\left(x_0 x_1 + y_0 y_1\right)\right\} A(x_0, y_0)\, E(x_0, y_0)\, dx_0\, dy_0$$

$E(x_0,y_0)$ = constant if a plane wave

This is just a Fourier Transform!

Interestingly, it's a Fourier Transform from position, x_0, to another position variable, x_1 (in another plane). Usually, the Fourier "conjugate variables" have reciprocal units (e.g., t & ω, or x & k). The conjugate variables here are really x_0 and $k_x = kx_1/z$, which have reciprocal units.

So the far-field light field is the Fourier Transform of the apertured field!

The Fraunhofer Diffraction formula

We can write this result in terms of the off-axis k-vector components:

$E(x,y)$ = const if a plane wave

$$E\left(k_x,k_y\right) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-i\left(k_x x + k_y y\right)\right] A(x,y)\, E(x,y)\, dx\, dy$$

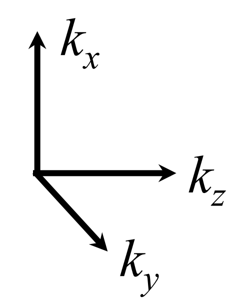
Aperture function

where we've dropped the subscripts, 0 and 1,

$$E\left(k_x,k_y\right) \propto \mathscr{F}\left\{A(x,y)\, E(x,y)\right\}$$

and:

$$k_x = k x_1/z \text{ and } k_y = k y_1/z$$



or:

$$q_x = k_x/k = x_1/z \text{ and } q_y = k_y/k = y_1/z$$

The Uncertainty Principle in Diffraction!

$$E\left(k_x, k_y\right) \propto \mathscr{F}\left\{A(x,y)\,E(x,y)\right\} \qquad k_x = k\,x_1/z$$

Because the diffraction pattern is the **Fourier transform** of the slit, there's an uncertainty principle between the slit width and diffraction pattern width!

If the input field is a plane wave and $\Delta x = \Delta x_0$ is the slit width,

$$\Delta x\,\Delta k_x > 1$$

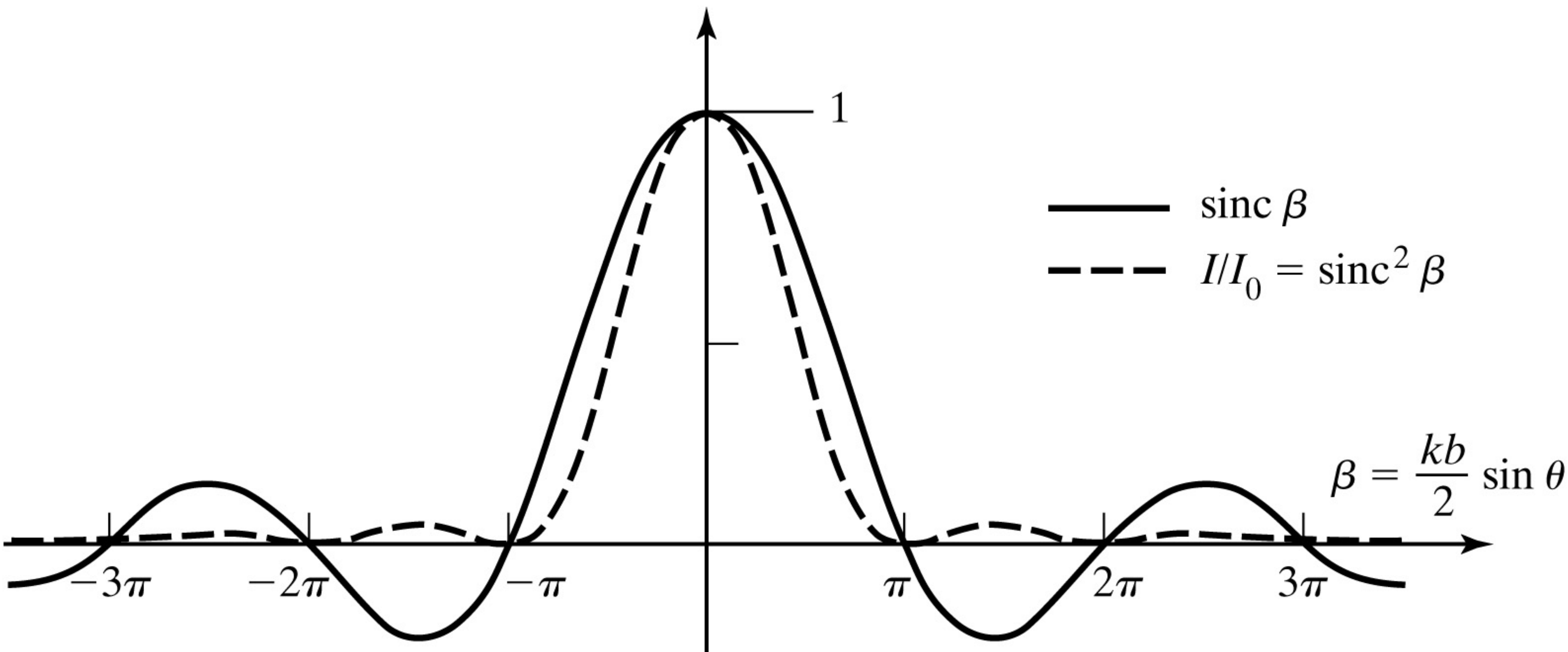
Or:

$$\Delta x_0\,\Delta x_1 > z/k$$

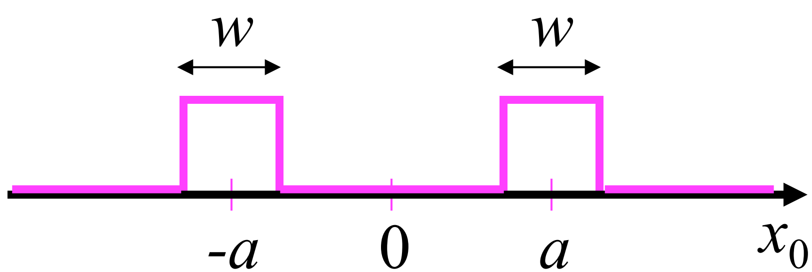
The smaller the slit, the larger the diffraction angle and the bigger the diffraction pattern!

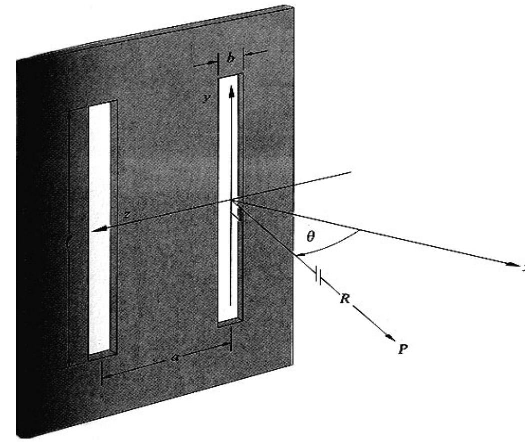
Fraunhofer Diffraction from a slit

Fraunhofer Diffraction from a slit is simply the Fourier Transform of a rect function, which is a sinc function. The irradiance is then sinc^2 .



Fraunhofer diffraction from two slits



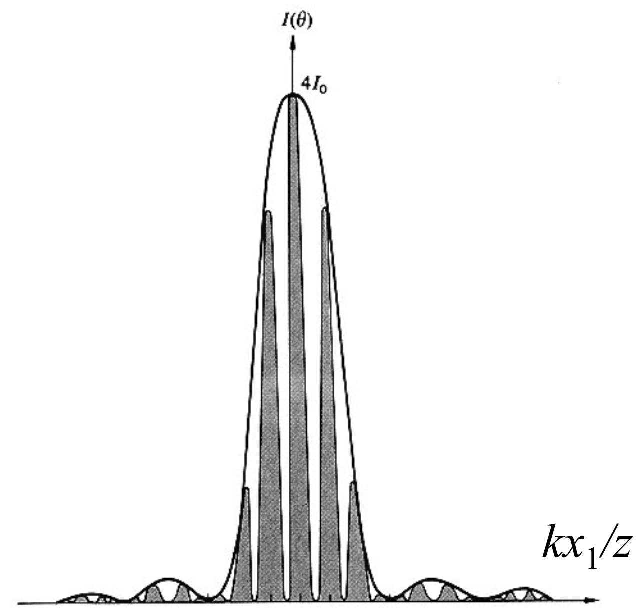


$$A(x_0) = \mathrm{rect}[(x_0+a)/w] + \mathrm{rect}[(x_0-a)/w]$$

$$E(x_1) \propto \mathscr{F}\{A(x_0)\}$$

$$\propto \mathrm{sinc}[w(kx_1/z)/2]\exp[+ia(kx_1/z)] + \mathrm{sinc}[w(kx_1/z)/2]\exp[-ia(kx_1/z)]$$

$$\boxed{E(x_1) \propto \mathrm{sinc}(wkx_1/2z)\cos(akx_1/z)}$$

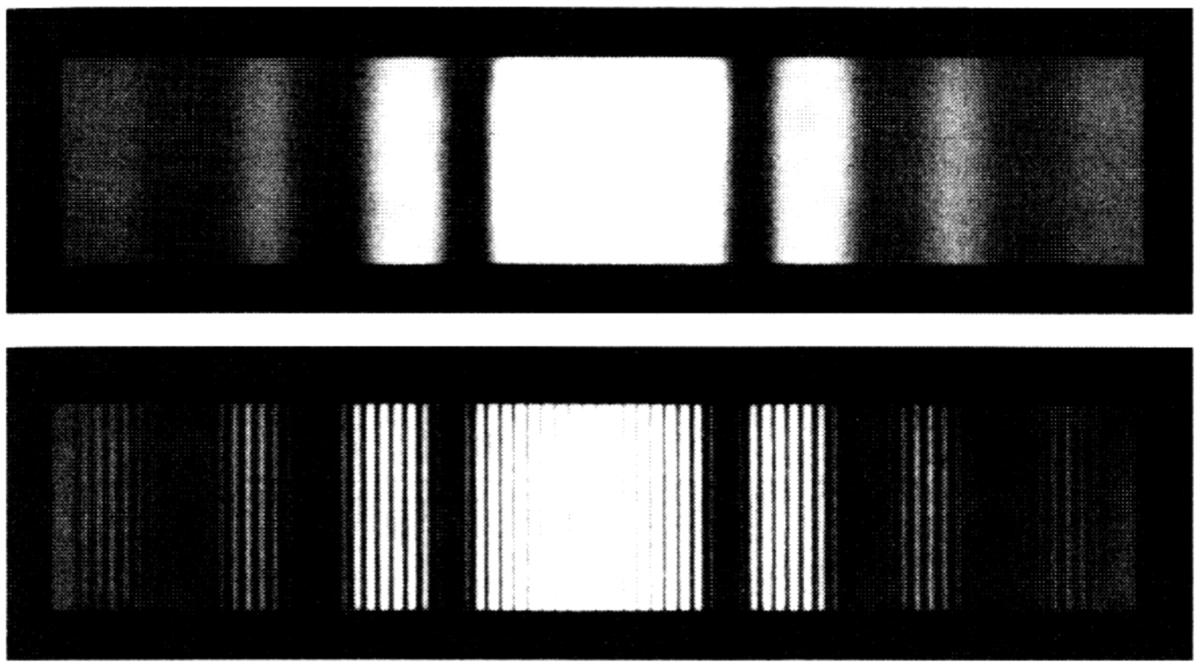


Diffraction from one- and two-slit screens

Fraunhofer diffraction patterns

One slit

Two slits



Gaussian Beam - Laser