Measurements & Uncertainty Analysis

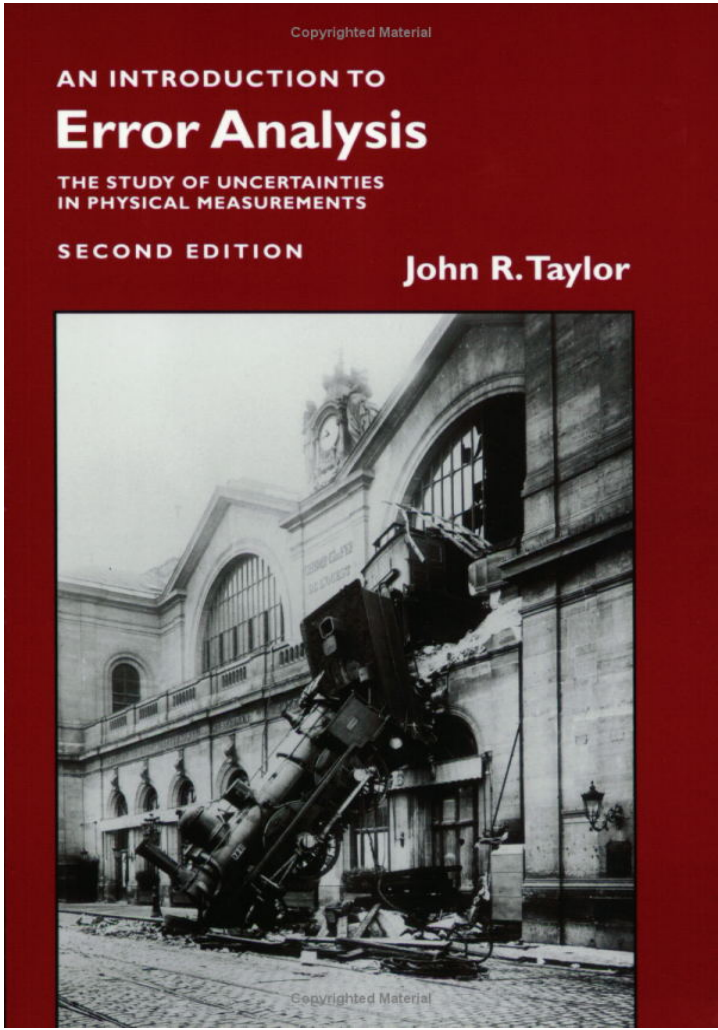
PHY451

September 3, 2014

MICHIGAN STATE
UNIVERSITY

References

- "An Introduction to Error Analysis, The Study of Uncertainties in physical measurements", 2nd edition, John. R. Taylor, 1997.
- http://www.lon-capa.org/~mmp/labs/error
- Class Website
  - http://www.pa.msu.edu/courses/PHY451/
  - Some materials pass word protected due to copyright issues
  - Pass word wuli451!



MICHIGAN STATE
UNIVERSITY

Accuracy, Precision, Error, Uncertainty

- **Accuracy:** Difference between measurements and true value (X_c)
- **Precision:** Spread in measurements (reproducibility, repeatability etc.)

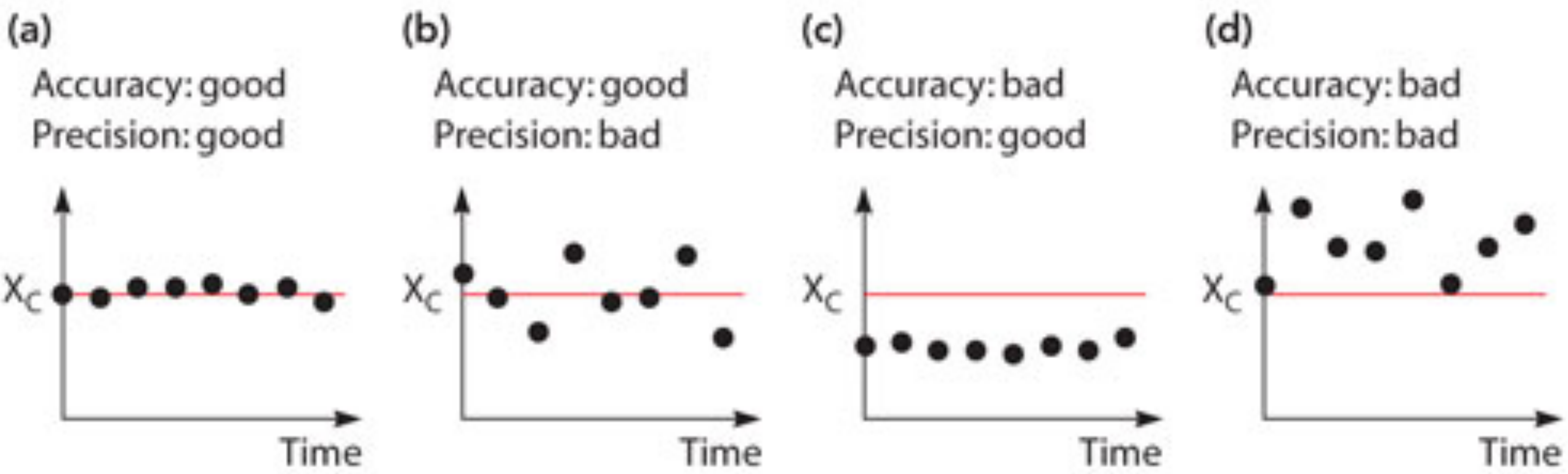


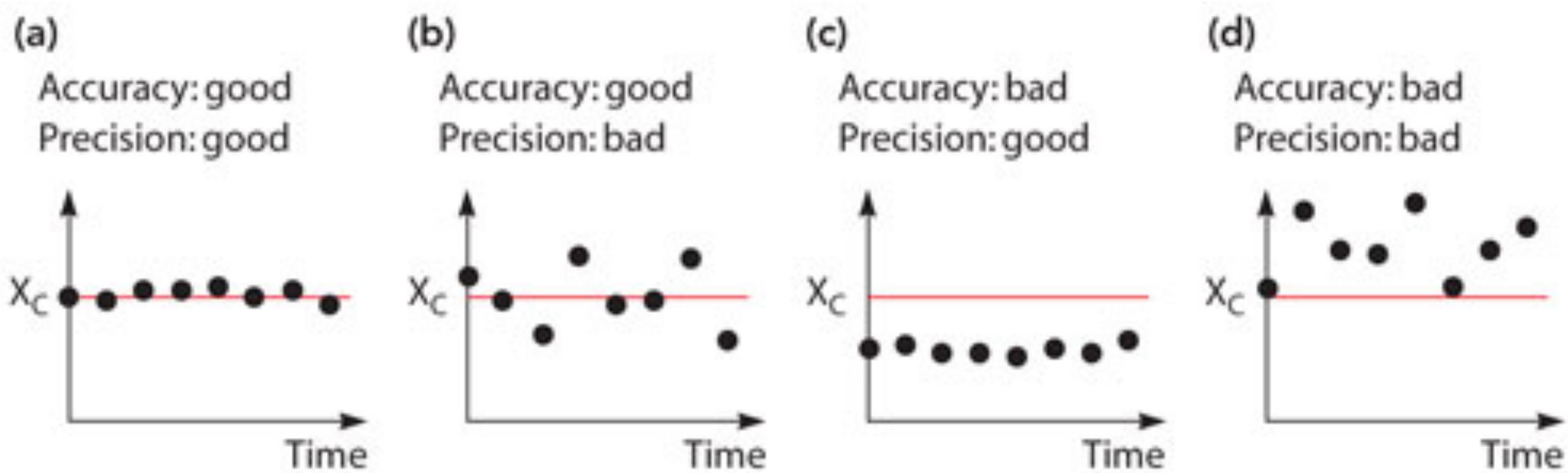
Figure from:
http://www.edn.com/design/test-and-measurement/4388718/Manage-your-measurement-errors

- **Error:** Accuracy but since don't usually know true value not used
- **Uncertainty:** Interval in which repeated measurements fall – usually used as does not require true value
  - **Random** uncertainty decreases precision
  - **Systematic** uncertainty decreases accuracy

MICHIGAN STATE UNIVERSITY

Uncertainty

- Uncertainty describes range over which true value likely to lie
  - Procedure (experiment) dependent
  - If again use same procedure under same conditions, should get new result but with difference between results similar to uncertainty
  - If use same procedure but under different conditions may or may not get result with difference between results similar to uncertainty
- Multiple measurements of same quantity can lead to distribution of measurements typically centered about average value and ranging in value above and below the average



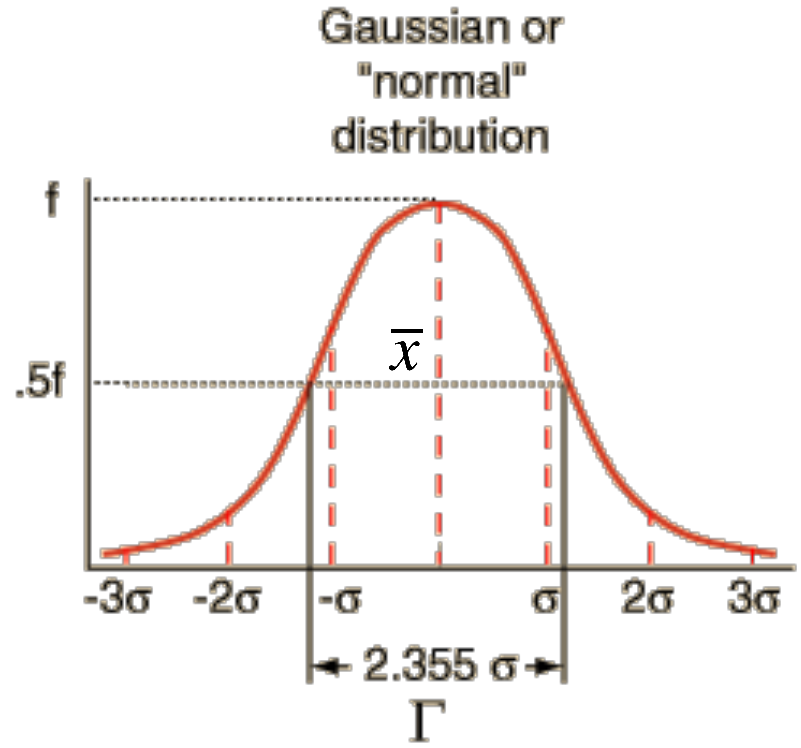
Distribution

- ***If*** **Gaussian (normal) distribution**
  - Standard deviation = σ_x
  - Average (mean) value $\bar{x}$
  - Uncertainty
    - ±1σ = 68.3% of measurements
    - ±2σ = 95.5% of measurements
    - ±3σ = 99.7% of measurements
    - Full width at half-maximum $\Gamma = 2.355\ \sigma_x$

$$f(x) = \frac{e^{-(x-\bar{x})^2/(2\sigma_x^2)}}{\sigma_x\sqrt{2\pi}}$$

- **Figure From:**

http://hyperphysics.phy-astr.gsu.edu/hbase/math/gaufcn2.html



MICHIGAN STATE UNIVERSITY

Uncertainty & Distribution

- If measurements follow Gaussian (normal) distribution, then quoted uncertainty related to confidence level – see Distribution slide
- E.g. if quote Length L = 9.6 m ± 0.2 m and measurements follow normal distribution then:
  - If 0.2m is 1 sigma – means 68.3% of values fall within (31.7% without)
  - If 0.2m is 2 sigma – means 95.5% of values fall within (4.5% without)
  - If 0.2m is 3 sigma – means 99.7% of values fall within (0.3% without)

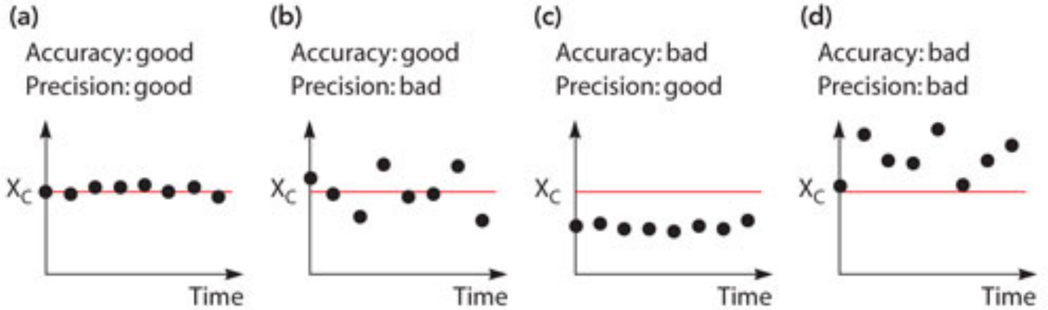
Measurements - 1

Measured values can vary due to a bias typically causing systematic uncertainty affecting all measurements in the same way so can be hard to detect – e.g.

» Ruler with incorrect scale can consistently give a reading too high or too low.
  - Solution may be to use multiple other "rulers" with identified "bad one" thrown out & replaced with "good one"
  - Data with "bad ruler" could be corrected if know in what way (too high or low and by how much) "bad ruler" deviated

» Meter can read systematically too high or too low but can have a calibration sequence that brings meter readings into compliance. Calibration may be lost over time and need to be consistently repeated.

- For values with spread or distribution on repetition of particular measurement - taking multiple measurements can reduce uncertainties – but won't reduce systematic uncertainty



MICHIGAN STATE
UNIVERSITY

Measurements - 2

- Measurements repeated <u>in same way</u> can identify variance which can be <u>due to e.g.</u>
  - External variables
    - » Earthquakes – can distort data during (movement) and/or after (misalignment) of apparatus
    - » Commercial radio – could distort data due to radio frequency noise but possibly only at specific times depending on station broad cast schedule
    - » Field distorting equipment introduced into experimental system like vibrating machinery near sensitive optical system could cause larger variance in data from before introduction but constant (variance) after or could cause systematic error
  - Internal variables - experimental equipment changing / failing – will effect data differently if short or long compared to data taking time – e.g. if piece of equipment fails over seconds can cause large variance in data set-to-data set (easier to discover) or e.g. if fails over weeks then can cause small variance in data set-to-data set (harder to discover) or even worse be intermittent (hardest to discover)
  - Experimental approach – if get consistent results (same mean / variance) then <u>infers</u> that random uncertainties are cause
    - » But does not rule out possibility of also having a systematic uncertainty

Measurements - 3

Measurements of same quantity in more than one way can help identify variance or bias or to otherwise confirm / reduce uncertainty in experimental result.

- **External variables to experiment**
  - » **Can work to control / test for external variables by maintaining / policing experimental area for offending elements**
  - » **Can develop methodologies less sensitive to external influences – e.g better mechanical / electrical isolation of experiment**
- **Internal variables to experiment**
  - » **Can develop test/calibration procedures to ensure hardware is operating properly**
  - » **Can develop alternative methodologies to test against systematic uncertainties, to reduce other uncertainties**
- **Experimental approaches – if get consistent results infers that have valid result to the degree that different experimental methodologies can provide independent approaches and not all likely to have uncertainties in same way**
  - » **Reduces but does not completely rule out possibility of systematic uncertainty**

Experiments - 1

- **Careful planning**
  - Understand experiment before start taking data. What is goal?
  - What measurements are required to achieve goal and how will these be obtained?
  - Be knowledgeable about other's prior efforts / approaches.
  - Consider alternatives to prior approaches.
  - Are there alternative / better methods to obtain / cross check these measurements?
  - Are there measurements more important (affect result more strongly) than others? Don't want to waste effort on improving measurement of parameter that is of little or no consequence to result. Focus on those with largest impact.
- **Careful execution**
  - Keep meticulous notebook - necessary approach for this course and important for your future endeavors whether industry or academia
  - Going completely through experiment "quickly" can provide early opportunity to see that can ***fully*** get necessary information for result and early identification of measurement problems, oversights, etc.
  - "Quick" pass provides better understanding so can enhance outcome of further more methodical experimentation
  - Methodical experimentation should involve cross checks to see if consistent and making "sense"

Experiments - 2

- **Data Analyses**
  - Understand analyses and how going to do it or you may miss data you need
  - Best to analyze / plot data as you go for early identification of missing or bad data
  - Think about publication and what figures / plots are necessary so have necessary data
  - Keep watch on sources of errors
  - Document process/procedure in lab notebook.
  - Some data sets too large for notebook representation - Be sure store / document location of data sets
  - Process data at least a cursory level as soon as possible to have early identification of possible issues, inconsistencies etc.

- **"Publication"**
  - Important to develop communication skills – presentations and papers of your experimental results provide good opportunities
  - Need to be able to explain clearly, concisely, and without bias (don't exaggerate or dramatically under rate your results)
    - » Explanation channel typically <u>both</u> in written document as well as oral presentations – <u>Both</u> skills important and <u>improve only with practice</u>
    - » Discussions with lab partners / colleagues helps bring issues/concepts in to focus providing better written and oral presentations

Measurement Result

State as Best Estimate ± Uncertainty or x ± δx – e.g.

- Single measurements
  - Use measured value for best estimate
  - For uncertainty - use accuracy of device e.g.
    - » Ruler marked in mm might have error of ~1mm ➔ ±0.5mm
    - » Micrometer might have error of ~0.1 mm ➔ ±0.05mm
- Multiple measurements
  - Use average value for best value
  - For uncertainty use standard deviation (defined later)

Measured quantities often used to calculate additional value – e.g.

- Area = length (L) x width (W)

If have measured L ± δL and W± δW, what is uncertainty in Area?

See Uncertainty Propagation

Simple Uncertainty Propagation - 1

- **Correlated Uncertainty**
  - Measured quantities added or subtracted ➔ uncertainties (δs) add
    - » Z = B + C – D, uncertainty in Z from uncertainties in B, C, & D
      - ➢ $\delta Z = \delta B + \delta C + \delta D$
  - Measured quantities multiplied or divided ➔ fractional uncertainties add
    - » Z = B•C/D, uncertainty in Z from uncertainties in B, C, & D
      - ➢ $\delta Z/ |Z| = \delta B/ |B| + \delta C/ |C| + \delta D/ |D|$

- **Uncorrelated Uncertainty**
  - Measured quantities added or subtracted ➔ uncertainties add in quadrature
    - » Z = B + C – D, uncertainty in Z from uncertainties in B, C, & D
      - ➢ $\delta Z = (\delta B^2 + \delta C^2 + \delta D^2)^{0.5}$
  - Measured quantities multiplied or divided ➔ fractional uncertainties add in quadrature
    - » Z = B•C/D, uncertainty in Z from uncertainty in B, C, & D
      - ➢ $\delta Z/ |Z|= ((\delta B/ |B|)^2 + (\delta C/ |C|)^2 + (\delta D/ |D|)^2)^{0.5}$

Simple Uncertainty Propagation - 2

- **Events occur randomly but with definite rate ➔ counting experiment – x counts**
  - **Uncertainty (standard deviation) from sqrt (counts) or** $\delta x = 1 \cdot \sigma_x = \sqrt{x}$
  - **Example – how many counts necessary to have 10% error with 95% confidence?**
    - **$2\sigma = 2\delta x = 2^*(x)^{0.5}$ = 95% confidence (see Distribution slide)**
    - **Then want 10% or $2^*(x)^{0.5} / x = 0.1$ ➔ x=400 counts**

- **Uncertainty with power function ➔ $Z = B^n$**
  - **$\delta Z / |Z| = n \bullet (\delta B / |B|)$**

Statistics - 1

- **Average, mean, 'typical value'** $\bar{x}$
  - **n measurements** x_i

$$\bar{x} = \frac{\sum_i x_i}{n}$$

- **Standard deviation, variance,** $\sigma_x = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{n-1}} = \sqrt{\langle x - \bar{x} \rangle}$

- **Result** $\bar{x} \quad \pm \quad \delta x$

$$\delta x = k \cdot \sigma_x, k = 1,2,3 \ldots$$

Statistics – 2

- **Function Z = f(u,v)**
  - **Variance**

$$\sigma_u = \sqrt{\frac{\sum_i (u_i - \bar{u})^2}{n-1}} = \sqrt{\langle u - \bar{u} \rangle}$$

$$\sigma_v = \sqrt{\frac{\sum_i (v_i - \bar{v})^2}{n-1}} = \sqrt{\langle v - \bar{v} \rangle}$$

  - **Covariance**

$$\sigma_{vu} = \sqrt{\frac{\sum_i (v_i - \bar{v})(u_i - \bar{u})}{n-1}} = \sqrt{\langle (v - \bar{v})(u_i - \bar{u}) \rangle}$$

General Uncertainty Propagation - 1

- **Function Z = f(u,v)**

$$\sigma_Z^2 = \sigma_u^2(\partial f/\partial u)^2 + \sigma_v^2(\partial f/\partial v)^2 + 2\sigma_{uv}^2(\partial f/\partial u)(\partial f/\partial v)$$

- **If uncorrelated ➔** $\sigma_{uv} = 0$
- **Example: Z = f(u,v) = a•u + b•v** $\sigma_Z^2 = \sigma_u^2 a^2 + \sigma_v^2 b^2 + 2\sigma_{uv}^2 a \cdot b$
  - **Uncorrelated ➔ add in quadrature ➔** $\sigma_Z^2 = \sigma_u^2 a^2 + \sigma_v^2 b^2$
  - **Correlated ➔ add ➔** $\sigma_Z^2 = (\sigma_u a + \sigma_v b)^2$
- **Example: Z = f(u,v) = a•u/v**

$$\sigma_Z^2 = \sigma_u^2(a/v)^2 + \sigma_w(-a \cdot u/v^2)^2 + 2\sigma_{uv}^2(-\frac{a^2 u}{v^3})$$

  - **Uncorrelated ➔ add fractionally in quadrature ➔** $\frac{\sigma_Z^2}{Z^2} = \frac{\sigma_u^2}{u^2} + \frac{\sigma_v^2}{v^2}$
  - **Correlated ➔ add fractionally ➔** $\frac{\sigma_Z}{Z} = \frac{\sigma_u}{u} + \frac{\sigma_v}{v}$

General Uncertainty Propagation - 2

- **Example: Z = f(u,v) = a•u^b^** $\sigma_Z = \sigma_u (a \cdot b \cdot u^{b-1})$ $\dfrac{\sigma_Z}{Z} = \dfrac{\sigma_u \cdot b}{u}$
- **Example: Z = f(u,v) = a•e^bu^** $\sigma_Z = \sigma_u (a \cdot b \cdot e^{bu})$ $\dfrac{\sigma_Z}{Z} = b \cdot \sigma_u$