

PHY422 Homework Set 11

1. [10 pts] Consider a particle of mass m , moving at angular momentum ℓ , under the influence of a central force of the form

$$F(r) = -\frac{\alpha}{r^2} - \frac{\beta}{r^4},$$

where $\alpha, \beta > 0$. (a) Find minimum ℓ for which circular orbits, $r = \text{const}$, are possible. How many of those orbits emerge depending on ℓ ? (b) Examine mathematically the stability of those orbits. (c) Sketch V_{eff} for the three cases: when no circular orbits are possible, only one and more are possible. (d) If only one circular orbit is possible, is it stable or unstable and why?

2. [5 pts] Johnson 6-1. Consider radial velocity $\frac{dr}{dt} = \frac{d\theta}{dt} \frac{dr}{d\theta}$. The time of travel from $r = a$ to $r = 0$ is $m a^2 / \ell$.
3. [5 pts] Johnson 6-2. Note: In this problem m is particle mass and μ is a parameter with dimension $\ell^2 / 2m$.

4. [5 pts] Johnson 6-4. The time is given by

$$t - \text{const} = \pm \frac{1}{2E} \sqrt{2m(E r^2 - \alpha) - \ell^2}.$$

5. [5 pts] Johnson 6-6. The period of motion follows from Eq. (6.5) in Johnson and it is

$$T = 2\pi \sqrt{\frac{R^3}{G(M+m)}}.$$

6. [10 pts] Consider the differential orbital equation for the variable $u = 1/r$ in the central-force problem.

- (a) First show that the orbital equation may be cast in the form

$$u'' = -u \left(1 + \frac{F}{F_{\text{cf}}}(u) \right),$$

where F_{cf} is centrifugal force.

- (b) Next turn to the dimensionless variable $v = r_0/r$, where r_0 is some starting distance away from origin, and assume an attractive power-law force of the form $F = -k/r^\alpha$, where $k > 0$. Show that the differential equation for v may be cast into the form

$$v'' = -v + \beta v^{\alpha-2},$$

where $\beta = \frac{|F|}{F_{\text{cf}}}$ is the ratio of the force magnitudes at the starting position where $v = 1$.

- (c) Next, use Mathematica or another math package to solve the orbital equation in v , assuming that r_0 is chosen an extremal distance from the origin. In the latter case, the starting conditions for v at $\theta = 0$ are $v(0) = 1$ and $v'(0) = 0$. Physically, the radius should decrease relative to the original position if $\beta > 1$ and increase if $\beta < 1$. Plot the orbits you arrive at over angular range that allows to illuminate orbital characteristics. Try out first the case of gravity with $\alpha = 2$, and $\beta > 1$, $\beta = 1$ and $\beta < 1$. Next turn to other values of α , such as $3/2$, $5/2$ and 3 . Comment on your results. Attach codes and plots to your homework, for a number of exemplary cases, or upload those to the Dropbox on Angel.

In Mathematica the commands might have the form:

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alpha = 2; beta = 1;
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Solution = NDSolve[{v''[theta] == -v[theta] + beta * v[theta]^(alpha-2), v'[0] == 0, v[0] == 1, rr0[theta] == 1/v[theta]}, {v, rr0}, {theta, 0, 28Pi}];
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PolarPlot[Evaluate[rr0[theta]/.Solution], {theta, 0, 28Pi}]
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