PHY820 Homework Set 11

1. [10 pts] Consider a particle of mass m, moving at angular momentum ℓ, under the influence of a central force of the form

$$F(r) = -\frac{\alpha}{r^2} - \frac{\beta}{r^4} \, ,$$

where $\alpha, \beta > 0$. (a) Find minimum ℓ for which circular orbits, $r = const$, are possible. How many of those orbits emerge depending on ℓ? (b) Examine mathematically the stability of those orbits. (c) Sketch V_{eff} for the three cases: when no circular orbits are possible, only one and more are possible. (d) If only one circular orbit is possible, is it stable or unstable and why?

2. [10 pts] (Goldstein)

   (a) Show that, if a particle follows a circular orbit under the influence of an attractive central force directed toward a fixed point on the circle, then the force varies as the inverse-fifth power of the distance from the point. Hint: Establish $r = r(\theta)$ and then either turn to $\dot{r}$ or to $\mathrm{d}^2 u/\mathrm{d}\,\theta^2$, where $u = 1/r$.

   (b) Show that for the orbit described above the total energy of the particle must be zero.

   (c) Find the period of the motion for the particle. Hint: Use the third Kepler's law.

3. [5 pts] Johnson 6-2. Note: In this problem m is particle mass and μ is a parameter with dimension $\ell^2/2m$.

4. [5 pts] Johnson 6-4. The time is given by

$$t - const = \pm \frac{1}{2E} \sqrt{2m(E\, r^2 - \alpha) - \ell^2} \, .$$

5. [5 pts] Johnson 6-6. The period of motion follows from Eq. (6.5) in Johnson and it is

$$T = 2\,\pi \sqrt{\frac{R^3}{G\,(M+m)}} \, .$$

6. [10 pts] Consider the differential orbital equation for the variable $u = 1/r$ in the central-force problem.

   (a) First show that the orbital equation may be cast in the form

$$u" = -u \left(1 + \frac{F}{F_{\text{cf}}}(u)\right) ,$$

   where F_{cf} is centrifugal force.

(b) Next turn to the dimensionless variable $v = r_0/r$, where r_0 is some starting distance away from origin, and assume an attractive power-law force of the form $F = -k/r^\alpha$, where $k > 0$. Show that the differential equation for v may be cast into the form

$$v" = -v + \beta\, v^{\alpha-2}\,,$$

where $\beta = \frac{|F|}{F_{\rm cf}}$ is the ratio of the force magnitudes at the starting position where $v = 1$.

(c) Next, use Mathematica or another math package to solve the orbital equation in v, assuming that r_0 is chosen an extremal distance from the origin. In the latter case, the starting conditions for v at $\theta = 0$ are $v(0) = 1$ and $v'(0) = 0$. Physically, the radius should decrease relative to the original position if $\beta > 1$ and increase if $\beta < 1$. Plot the orbits you arrive at over angular range that allows to illuminate orbital characteristics. Try out first the case of gravity with $\alpha = 2$, and $\beta > 1$, $\beta = 1$ and $\beta < 1$. Next turn to other values of α, such as $3/2$, $5/2$ and 3. Comment on your results. Attach codes and plots to your homework, for a number of exemplary cases, or upload those to the Dropbox on Angel.

In Mathematica the commands might have the form:

```
alpha = 2; beta =1;
Solution = NDSolve[{v"[theta] == -v[theta] + beta * v[theta]^(alpha-2), v'[0] == 0, v[0] == 1, rr0[theta] == 1/v[theta]}, {v, rr0}, {theta, 0, 28Pi}];
PolarPlot[Evaluate[rr0[theta]/.Solution], {theta, 0, 28Pi}]
```