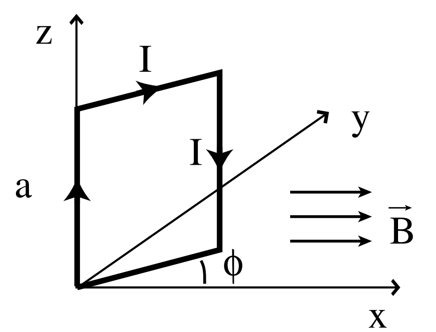
PHY820 Homework Set 14

1. [5 points] (Goldstein) The Lagrangian for a system can be written as

$$L = a\,\dot{x}^2 + b\,\frac{\dot{y}}{x} + c\,\dot{x}\,\dot{y} + f\,y^2\,\dot{x}\,\dot{z} + g\,\dot{y} - k\,\sqrt{x^2+y^2}\,,$$

where a, b, c, f, g, and k are constants. What is the Hamiltonian? What quantities are conserved?

2. [10 pts] An ideally conductive square loop can rotate around its side placed on the z-axis, as shown, within a constant uniform magnetic field $\vec{B}$ along the x-axis. The loop's side length is a, moment of inertia is J and self-inductance is $\mathcal{L}$. As generalized coordinates describing the loop, one can use the angle ϕ of the loop relative to the x-axis and the net charge q that passed around the loop in the clockwise direction. (The current is $I \equiv \dot{q}$.) In terms of these coordinates, the Lagrangian for the loop can be written as



$$L(\phi,\dot{\phi},q,\dot{q}) = \frac{1}{2}\,J\,\dot{\phi}^2 + \frac{1}{2}\,\mathcal{L}\,\dot{q}^2 - \dot{q}\,a^2\,B\,\sin\phi\,.$$

Here, one can recognize the rotational and inductive energies of the loop and an interaction term of the loop's magnetic moment with the field. (a) From the Lagrangian, find the conserved quantities for the motion of the loop. Can you interpret those quantities? (b) Obtain a Hamiltonian for the loop in terms of the specified coordinates and generalized momenta. (c) Exploit the conservation laws from (a) to obtain an effective potential $U_{eff}(\phi)$ for the motion of the loop in ϕ. Sketch the potential and discuss qualitatively the possible motions in ϕ depending on initial conditions.

3. [5 points] (Goldstein)

   (a) The Lagrangian for a system of one degree of freedom can be written as

$$L = \frac{m}{2}\left(\dot{q}^2\sin^2\omega t + \dot{q}\,q\,\omega\,\sin 2\omega t + q^2\,\omega^2\right).$$

   What is the corresponding Hamiltonian? Is it conserved?

   (b) Introduce a new coordinate defined by

$$Q = q\sin\omega t\,.$$

   Find the Lagrangian in terms of the new coordinate and the corresponding Hamiltonian. Is H conserved?

4. [5 points] (Goldstein) Show from the Poisson bracket condition for conserved quantities that the Runge-Lenz vector $\boldsymbol{A}$,

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - \frac{m\,k\,\boldsymbol{r}}{r}\,.$$

is a constant of the motion for the Kepler problem.

5. [5 pts] A canonical transformation, representing the rotation by an angle α in the phase-space, is given by the equations

$$Q = q\cos\alpha + \lambda\,p\sin\alpha\,, \qquad P = p\cos\alpha - \frac{1}{\lambda}\,q\sin\alpha\,,$$

where λ is some scale parameter. (a) Find the equations for an inverse transformation. (b) Obtain $p = p(q, P)$ and $Q = Q(q, P)$. (c) Determine the generating function $F(q, P)$ for the above canonical transformation.

6. [10 pts] (Goldstein)

   (a) Show directly that the transformation

$$Q = \log\left(\frac{1}{q}\sin p\right), \qquad P = q\,\cot p\,,$$

   is canonical.

   (b) For the point transformation in a system of two degrees of freedom,

$$Q_1 = q_1^2\,, \qquad Q_2 = q_1 + q_2\,,$$

   find the most general transformation equations for P_1 and P_2 consistent with the overall transformation being canonical. Show that with a particular choice for P_1 and P_2 the Hamiltonian

$$H = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2$$

   can be transformed to one in which both Q_1 and Q_2 are ignorable. By this means solve the problem and obtain expressions for q_1, q_2, p_1, and p_2 as functions of time and their initial values.

7. [10 pts] Consider the damped Mathieu equation

$$\ddot{x} = -[1 + \epsilon\cos(\omega_m\,t)]\,x - \beta\,\dot{x}\,.$$

   (a) What is the angular frequency of oscillations ω_0 in the absence of damping and modulation of the spring constant, i.e. $\beta = 0$ and $\epsilon = 0$? What is the angular frequency of oscillations ω in the presence of damping but lack of modulation, i.e. $\epsilon = 0$, but $\beta \neq 0$?

(b) Solve the Mathieu equation numerically using a math package, such as Mathematica. Show that for weak damping, such as $\beta \sim 0.02$, and moderate modulation, such as $|\epsilon| \sim 1/3$ (under any circumstances use only $|\epsilon| < 1$), the amplitude of oscillations may increase indiscriminately with time at the frequencies of modulation close to the resonance frequencies of $\omega_m \lesssim 2\,\omega/n$. Here n, is integer and the growth tends to be particularly pronounced at low n and especially $n = 1$. *In solving the equation, you can modify your code from set 5 of the homework.* Attach the plots together with the printout of your code to your homework or upload them to the Dropbox on Angel.