PHY820 Homework Set 3

1. [10 pts] (Goldstein) Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric friction, is

$$m \frac{\mathrm{d}v}{\mathrm{d}t} = -v' \frac{\mathrm{d}m}{\mathrm{d}t} - m\, g \,,$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 2.1 km/s and a mass per second equal to $l/60$th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel lo the weight of the empty rocket must be almost 300!

2. [5 pts] Johnson, problem 2.6. The proper result is

$$T = 2\sqrt{\frac{2m}{a}}\, \tan^{-1}\left\{\frac{2}{b}\sqrt{aE}\right\},$$

and other representations of that result are possible, including one close that in the textbook.

3. [5 pts] Johnson, problem 2.7.

4. [10 pts] Johnson, problem 2.9. The relation that the separatrix satisfies is

$$2\sqrt{\frac{2V_0}{m}}\frac{t}{a^2} = \frac{x}{a}\sqrt{1+\frac{x^2}{a^2}} + \sinh^{-1}\frac{x}{a}\,.$$

5. [5 pts] Johnson, problem 2.11.

6. [5 pts] Based on Johnson, problem 2.20: A particle is moving in the x direction subject to the following differential equation

$$\ddot{x} = -\beta\, \dot{x}\left(\dot{x}^2 + \omega^2 x^2 - A^2\right) - \omega^2 x\,,$$

where β, A and ω are positive parameters. In addition to a linear restoring force, there is a velocity-dependent force that may either accelerate or decelerate the particle. The model is an extension of simple harmonic motion which occurs for $\beta = 0$.

Show that the positive quantity

$$E(t) = \dot{x}^2(t) + \omega^2 x^2(t)$$

satisfies the relation

$$\dot{E}(t) = -2\beta\, \dot{x}^2\left(E(t) - A^2\right).$$

Show that $E(t)$ decreases with time whenever $E(t) > A^2$, and that $E(t)$ increases with time whenever $E(t) < A^2$. Also, note that when $E(t) = A^2$, x must depend on time. In fact, the curve

$$\dot{x}^2(t) + \omega^2\, x^2(t) = A^2$$

represents the limiting behavior of $x(t)$ in the limit $t \to \infty$.

The parameter ω is the limiting frequency of oscillation and A/ω is the limiting amplitude, whereas β determines the rate of approach to the limit cycle.