Special Relativity (Einstein, 1905)

(1) We can use any inertial coordinate system.

(2) The equations of physics are covariant --- they take the same form for all inertial frames.

(3) For example, the equations of electromagnetism:

Maxwell's equations

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$$

$$\partial_\nu G^{\mu\nu} = 0 \text{ where } G^{\mu\nu} = \tfrac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

Conservation of energy and momentum:

$$\mu_0 T^{\mu\nu} = F^{\mu\rho} F^{\nu}{}_{\rho} - \tfrac{1}{4}\,\eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

$$\partial_\nu T^{\mu\nu} = -F^{\mu\rho} J_\rho$$

(4) The metric tensor is "euclidean"

$$\eta_{\mu\nu} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

General Relativity (Einstein, 1916)

(1) We can use any coordinate system.

(2) The equations of physics are covariant --- they take the same form for all coordinate systems.

(3) The metric tensor, $g_{\mu\nu}(\underline{\boldsymbol{x}})$, may be non-euclidean.

(4) Gravity is curvature.

(5) The Einstein field equation

$$R^{\mu\nu} - \tfrac{1}{2}\, g^{\mu\nu} g_{\alpha\beta} R^{\alpha\beta} = (8\pi G/c^4)\; T^{\mu\nu}$$

$R^{\mu\nu}$ = Ricci curvature tensor

$T^{\mu\nu}$ = stress energy tensor of the gravitational source

The motion of particles in free fall

("Free fall" means that the only force on the particle is gravity.)

Postulate: The particle moves on a geodesic curve.

The geodesic equation is

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}$$

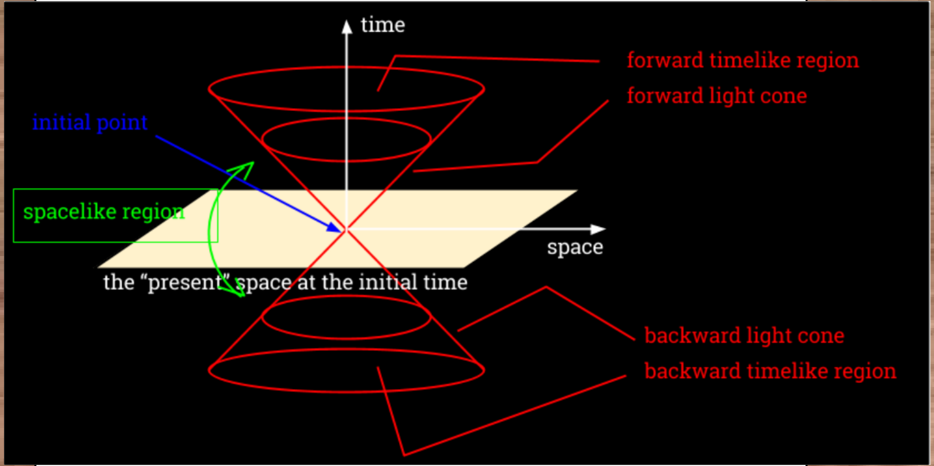
$\Gamma^\mu_{\rho\sigma}$ = the Christoffel symbols; $\Gamma^\mu_{\rho\sigma} = \Gamma^\mu_{\sigma\rho}$

A light ray moves on a ***null*** geodesic.

The purpose of today's lecture: derive the geodesic equation.

The light cone

Picture of a local inertial frame of reference…



A massive particle can only move inside the forward light cone. For example, a free particle has v < c.

A light ray (wave packet or photon) moves on the forward light cone;
$|\delta \mathbf{x}| / \delta t = c$.

Particle in free fall in G. R.

The proper time for the particle to move from A to B along a curve Γ is

$$c\tau(\Gamma) = \int c\,d\tau = \int \sqrt{-g_{\mu\nu}\,dx^\mu dx^\nu}$$

Parametrize the curve by some parameter σ; i.e., $x^\mu = x^\mu(\sigma)$ are the points on the curve; σ = 0 is A and σ = 1 is B. Then

$$c\tau(\Gamma) = \int_0^1 \sqrt{-g_{\mu\nu}(x)\,\frac{dx^\mu}{d\sigma}\,\frac{dx^\nu}{d\sigma}}\;d\sigma$$

Definition: The ***geodesic*** curve from A to B is the curve with minimum proper time.

Let $x^\mu(\sigma)$ be the geodesic curve from A to B; then consider infinitesimal changes of the curve, $x^\mu(\sigma) \to x^\mu(\sigma) + \delta x^\mu(\sigma)$. We must have $\delta\tau = 0$ for any change $\delta x^\mu(\sigma)$.

Definition. For a geodesic curve Γ, $\delta\tau = 0$ for any change of Γ (with endpoints fixed).

Analogy to classical mechanics - the Euler Lagrange equation

$S = \int_a^b L(\,dx/dt, x)\,dt$

(action and lagrangian)

$\delta S = 0$

$\Rightarrow\ d/dt\,(\partial L/\partial x') - \partial L/\partial x = 0$

(Lagrange's equation)

Replace t by σ ; replace x(t) by $x^\alpha(\sigma)$.

We have

$$L\left(\frac{dx^\mu}{d\sigma}, x^\alpha\right) = \sqrt{-g_{\mu\nu}(x)\,\frac{dx^\mu}{d\sigma}\,\frac{dx^\nu}{d\sigma}}$$

so $\delta\tau = 0$ implies

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0$$

$\dot{x}^\alpha \equiv \dfrac{dx^\alpha}{d\sigma}$

← 4 equations

Calculations - be very careful about the indices!

$$L = \sqrt{-g_{\mu\nu}(x)\,\dot{x}^\mu\,\dot{x}^\nu} = \cancel{m}\sqrt{-g\dot{x}\dot{x}}$$

① $$\frac{\partial L}{\partial x^\alpha} = \frac{1}{2}(-g\dot{x}\dot{x})^{-1/2}(-)\,\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu\dot{x}^\nu$$

Note : $c\,d\tau = \sqrt{-g\dot{x}\dot{x}}\;d\sigma$

so $\sqrt{-g\dot{x}\dot{x}} = c\,\frac{d\tau}{d\sigma}$

$$\frac{\partial L}{\partial x^\alpha} = -\frac{1}{2c}\,\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\,\frac{d\sigma}{d\tau}\,\frac{dx^\mu}{d\sigma}\,\frac{dx^\nu}{d\sigma}$$
$$= \frac{-1}{2c}\,\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau}\,\frac{d\tau}{d\sigma}$$

We first used σ to parametrize points along the curve; but now we'll use ***τ*** (proper time) to parametrize the curve.

② $$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{1}{2}(-g\dot{x}\dot{x})^{-1/2}(-)\left\{g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right\}$$

equal b/c $g_{\mu\alpha} = g_{\alpha\mu}$

$$= -g_{\alpha\nu}\,\frac{dx^\nu}{d\sigma}\,\frac{d\sigma}{c\,d\tau}$$
$$= -\frac{1}{c}\,g_{\alpha\nu}\,\frac{dx^\nu}{d\tau}$$

③ $$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = \frac{d\tau}{d\sigma}\,\frac{d}{d\tau}\left(-\frac{1}{c}\,g_{\alpha\nu}(x)\,\frac{dx^\nu}{d\tau}\right)$$
$$= \frac{-1}{c}\,\frac{d\tau}{d\sigma}\left\{\frac{\partial g_{\alpha\nu}}{\partial x^\mu}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau} + g_{\alpha\nu}(x)\,\frac{d^2x^\nu}{d\tau^2}\right\}$$

④ ③ = ① ⇒

$$g_{\alpha\nu}\,\frac{d^2x^\nu}{d\tau^2} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau} = \frac{1}{2}\,\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau}$$

I can replace this by its symmetric part

$$\frac{1}{2}\frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{1}{2}\frac{\partial g_{\alpha\mu}}{\partial x^\nu}$$

Notation for raising and lowering indices:

The metric tensor is $g_{\mu\nu}$.

The inverse of the metric tensor is $g^{\mu\nu}$.

That is, $g^{\mu\alpha}\, g_{\alpha\nu} = \delta_{\mu\nu}$.

Now multiply both sides of the equation by $g^{\mu\alpha}$.

$$g_{\alpha\nu} \frac{d^2x^\nu}{d\tau^2} = \frac{-1}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{\partial g_{\alpha\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

$$\frac{d^2x^\mu}{d\tau^2} = -\frac{1}{2} g^{\mu\alpha} \left(\frac{\partial g_{\alpha\rho}}{\partial x^\sigma} + \frac{\partial g_{\alpha\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\sigma}}{\partial x^\alpha} \right) \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}$$

Geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu{}_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}$$

The Christoffel symbols

$$\Gamma^\mu{}_{\rho\sigma} = \frac{1}{2} g^{\mu\alpha} \left(\frac{\partial g_{\alpha\rho}}{\partial x^\sigma} + \frac{\partial g_{\alpha\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\sigma}}{\partial x^\alpha} \right)$$

Corollary.
If there is no gravity then the particle moves on a straight line. (Or, for a euclidean space, the geodesics are straight lines.)

Proof.
If there is no gravity then the coordinate system is an inertial frame of reference:

$g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\mu{}_{\rho\sigma} = 0$

$d^2x^\mu / d\tau^2 = 0$ and $dx^\mu / d\tau = \kappa^\mu$ (constant)

Note that $-g_{\mu\nu}\, dx^\mu\, dx^\nu = c^2 (d\tau)^2$ (proper time)

$-\eta_{\mu\nu}\, dx^\mu\, dx^\nu = \{ (\kappa^0)^2 - \boldsymbol{\kappa}^2 \} (d\tau)^2 = c^2 (d\tau)^2$

so $(\kappa^0)^2 - \boldsymbol{\kappa}^2 = c^2$.

Thus the velocity is constant, and less than c.

(Exercise: calculate $d\mathbf{x}/dt$.)

Next time:

The metric tensor for spacetime outside a spherical mass is the Schwarzschild metric.

How does a light ray move in the neighborhood of a spherical mass?

- ➢ deflection of light by the sun; an important historical test of the theory of general relativity.

- ➢ light cannot escape from a black hole if it comes too close (classically)