Motion of particles and light rays in the neighborhood of a spherical mass

The Schwarzschild metric

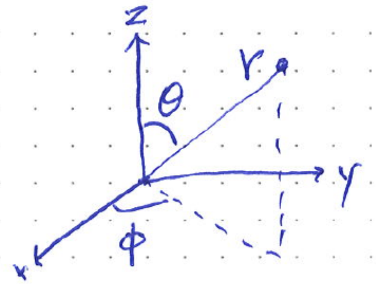
For empty space in the neighborhood of a spherical mass, the invariant distance between two events is

$$c^2(d\tau)^2 = \left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta(d\phi)^2$$

($d\tau$ = proper time for time like separations)

We're using these coordinates: $\{t, r, \theta, \varphi\}$

(polar coordinates for the 3D space at a constant time)



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

We're interested in the trajectory for 4D spacetime.

Compare Minkowski space (mass M = 0)

$$c^2(d\tau)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

$$dx = dr\sin\theta\cos\phi + r\cos\theta\,d\theta\cos\phi - r\sin\theta\sin\phi\,d\phi$$
$$dy = dr\sin\theta\sin\phi + r\cos\theta\,d\theta\sin\phi + r\sin\theta\cos\phi\,d\phi$$
$$dz = dr\cos\theta - r\sin\theta\,d\theta$$

$$c^2(d\tau)^2 = c^2(dt)^2 - (dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta(d\varphi)^2$$

The Schwarzschild metric tensor

$$c^2(d\tau)^2 = -\, g_{\mu\nu}\, dx^\mu\, dx^\nu$$

i.e., $x^0 = ct$; $x^1 = r$; $x^2 = \theta$; $x^3 = \varphi$.

$$g_{\mu\nu} = \begin{array}{c|cccc} {}_{\mu}\backslash^{\nu} & t & r & \theta & \phi \\ \hline t & -\Sigma(r) & 0 & 0 & 0 \\ r & 0 & 1/\Sigma(r) & 0 & 0 \\ \theta & 0 & 0 & r^2 & 0 \\ \phi & 0 & 0 & 0 & r^2\sin^2\theta \end{array}$$

$$\Sigma(r) = 1 - 2GM/c^2 r$$

A particle moves on a geodesic; i.e., on a curve Γ such that $\delta\tau = 0$ for any change of Γ.
A light ray (wave packet or photon) travels on a null geodesic; i.e., a geodesic with $\tau = 0$.

Now we could use the geodesic equation,

$$\frac{d^2 x^\mu}{d\tau^2} = -\,\Gamma^\mu{}_{\alpha\beta}\,\frac{dx^\alpha}{d\tau}\,\frac{dx^\beta}{d\tau}$$

but then we would need to calculate the Christoffel symbols
$\Gamma^\mu \alpha\beta$ $\{\ \mu, \alpha, \beta\ = t,\ r,\ \theta,\ \varphi\}$
Instead, it's easier to go back to first principles. Require $\delta\tau = 0$ where

$$c\,\tau(\Gamma) = \int_0^1 \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}\; d\sigma$$

$\Rightarrow$ the Euler-Lagrange equations

$$d/d\sigma\,(\partial L/\partial \xi') - \partial L/\partial \xi = 0$$

for all 4 independent variations $\delta\xi = \delta t,\ \delta r,\ \delta\theta,\ \delta\varphi$.

Euler-Lagrange equations

$d/d\sigma\,(\partial L/\partial \xi') - \partial L/\partial \xi = 0$ for $\xi = t,\ r,\ \theta,\ \varphi$;

with $L = (-g_{\mu\nu}\, x'^{\mu}\, x'^{\nu})^{1/2}$

{ x'^{μ} means $dx^{\mu}/d\sigma$ }

Note:

$$c\,\delta\tau = \sqrt{-g_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma}}\,d\sigma$$

$$\sqrt{-g\,\dot{x}\,\dot{x}} = c\,d\tau/d\sigma$$

Case of δt (or, δx^0 where $x^0 = ct$)

$$0 = \frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^0}\right) - \frac{\partial L}{\partial x^0}$$

$$= \frac{d}{d\sigma}\left\{\frac{1}{2}(-g\dot{x}\dot{x})^{-1/2}(-g_{00})\,2\frac{dx^0}{d\sigma}\right\} - 0$$

$$= \frac{d}{d\sigma}\left\{\Sigma(r)\,\frac{dt}{d\tau}\right\} \qquad \boxed{g_{00} = -\Sigma(r)}$$

Result $\quad \Sigma(r)\,\frac{dt}{d\tau} = \alpha$, a constant. ①

This constant of the motion is related to ***energy;*** the metric is invariant with respect to translations in time.

Case of δφ

$$0 = \frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi}$$

$$= \frac{d}{d\sigma}\left\{\frac{1}{2}(-g\dot{x}\dot{x})^{-1/2}(-g_{\phi\phi})\,2\frac{d\phi}{d\sigma}\right\} - 0$$

$$= \frac{d}{d\sigma}\left\{-r^2\sin^2(\theta)\,\frac{d\phi}{d\tau}\right\} \qquad \boxed{g_{\phi\phi} = r^2\sin^2\theta}$$

Result $\quad r^2\sin^2\theta\,\frac{d\phi}{d\tau} = \lambda$, a constant. ②

This constant of the motion is related to ***angular momentum;*** the metric is invariant with respect to translations in φ .

Case of $\delta\theta$

$$0 = \frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta}$$

$$= \frac{d}{d\sigma}\left\{\frac{1}{2}(-g\dot x\dot x)^{-1/2}(-g_{\theta\theta})\,2\frac{d\theta}{d\sigma}\right\}$$

$$-\frac{1}{2}(-g\dot x\dot x)^{-1/2}\left(-\frac{\partial g_{\phi\phi}}{\partial\theta}\right)\left(\frac{d\phi}{d\sigma}\right)^2$$

$$0 = (-g\dot x\dot x)^{1/2}\ \frac{d}{d\tau}\left\{-r^2\frac{d\theta}{d\tau}\right\}$$

$$+\frac{1}{2}(-g\dot x\dot x)^{1/2}\ 2r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2$$

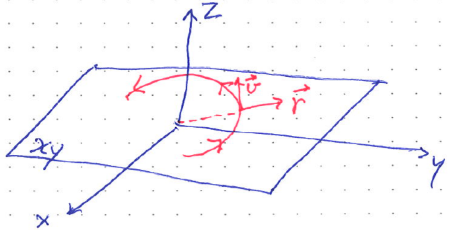
$$0 = \frac{d}{d\tau}\left(-r^2\frac{d\theta}{d\tau}\right) + r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2$$

We have a solution for this equation.

$\theta = \pi/2$

Why?

The orbit will lie in a plane, because angular momentum is conserved. Choose the orbit plane to be the xy plane. All points in the xy plane have $\theta = \pi/2$.
(Newtonian: $\mathbf{L} = \mathbf{r} \times m\mathbf{v} = mrv\sin(\theta(\mathbf{r},\mathbf{v}))\,\mathbf{e}_z$
$\mathbf{L}$ is always parallel to the z-axis, so $\mathbf{r}$ and $\mathbf{v}$ are always in the xy plane.)



Now go back and put $\theta = \pi/2$ into the equations.

$$\Sigma(r)\frac{dt}{d\tau} = \alpha \qquad (1)$$

$$r^2\frac{d\phi}{d\tau} = \lambda \qquad (2)$$

Case of δr

We don't need to derive the Euler-Lagrange equation for r; instead, we can just use this:

$$g_{rr} = \frac{1}{\Sigma(r)}$$

$$c^2(d\tau)^2 = -g_{\mu\nu}\,dx^\mu dx^\nu$$

$$= -g_{00}(c\,dt)^2 - g_{rr}(dr)^2 - g_{\theta\theta}(d\theta)^2 - g_{\phi\phi}(d\phi)^2$$

$$= \Sigma c^2\left(\frac{\alpha\,d\tau}{\Sigma}\right)^2 - \frac{1}{\Sigma}\left(\frac{dr}{\cancel{\Sigma}}\right)^2 - r^2(d\theta)^2 - r^2\sin^2\theta\left(\frac{\lambda\,d\tau}{r^2}\right)^2$$

$$\theta = \pi/2 \;;\; d\theta = 0$$

$$= \frac{\alpha^2}{\Sigma}c^2(d\tau)^2 - \frac{(dr)^2}{\Sigma} - \frac{\lambda^2}{r^2}(d\tau)^2$$

Result

$$\left(\frac{dr}{d\tau}\right)^2 = (\alpha^2 - \Sigma)c^2 - \frac{\lambda^2}{r^2}\Sigma \qquad \textbf{(3)}$$

The orbit curve: what is r(φ) ?

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{(dr/d\tau)^2}{(d\phi/d\tau)^2} = \frac{(\alpha^2-\Sigma)c^2 - \lambda^2\Sigma/r^2}{(\lambda/r^2)^2}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{\lambda^2}\left\{(\alpha^2-\Sigma)c^2 - \frac{\lambda^2}{r^2}\Sigma\right\}$$

$$\Sigma(r) = 1 - \frac{2GM}{c^2 r}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{(\alpha^2-1)c^2}{\lambda^2}r^4 + \frac{2GM}{\lambda^2}r^3 - r^2 + \frac{2GM}{c^2}r$$

Newtonian mechanics for planetary motion

Energy is conserved and angular momentum is conserved.

Newtonian Mechanics for planetary motion

Energy is conserved,

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} = E = m\mathcal{E};$$

and angular momentum is conserved

$$mr^2\frac{d\phi}{dt} = L = m\lambda.$$

So, we have

$$\dot{r}^2 = 2\mathcal{E} - \frac{\lambda^2}{r^2} + \frac{2GM}{r}$$

and $\dot{\phi}^2 = \left(\frac{\lambda}{r^2}\right)^2$.

Thus the orbit equation is

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\dot{r}^2}{\dot{\phi}^2} = \frac{r^4}{\lambda^2}\left\{2\mathcal{E} - \frac{\lambda^2}{r^2} + \frac{2GM}{r}\right\}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{2\mathcal{E}}{\lambda^2}r^4 - r^2 + \frac{2GM}{\lambda^2}r^3$$

Compare the orbit equation from Newtonian mechanics, to the orbit equation from General Relativity.

Compare the equation from G.R.:

Let $2\mathcal{E} = (\alpha^2-1)c^2$. Then for G.R.,

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{2\mathcal{E}}{\lambda^2}r^4 + \frac{2GM}{\lambda^2}r^3 - r^2 + \frac{2GMr}{c^2}$$

For Newtonian Mechanics, the orbits are conic sections (hyperbola, parabola, or ellipse). The relativity perturbation $= 2GMr/c^2$.

Einstein's calculation of the precession of the perihelion of mercury, as a test of the theory of general relativity.

Next time: motion of light rays