**Motion of light rays in the neighborhood of a spherical mass**

**Review**

*The Schwarzschild metric*

$$c^2(d\tau)^2 = \Sigma(r)\, c^2(dt)^2 - (dr)^2/\Sigma(r) - r^2\,(d\theta)^2 - r^2 \sin^2\theta\,(d\varphi)^2 \;;$$

$\Sigma(r) = 1 - 2GM/c^2 r = 1 - r_S\, r$ ;

$r_S$ = Schwarzschild radius = $2GM/c^2$

*Geodesic equations for coordinates* $\{t, r, \theta, \varphi\}$

$$\Sigma(r)\, dt/d\tau = \alpha \qquad (1)$$

$$r^2\, d\varphi/d\tau = \lambda \qquad (2)$$

$$\theta = \pi/2 \qquad (3)$$

$$(dr/d\tau)^2 = [\,\alpha^2 - \Sigma(r)\,]\, c^2 - \lambda^2\, \Sigma(r)/r^2 \qquad (4)$$

Now, light rays (wave packets or photons) travel on ***null*** geodesics. A null geodesic is a geodesic curve for which $d\tau = 0$. Light travels on the light cone; in a locally inertial frame,

$$c^2(d\tau)^2 = c^2(dt')^2 - (d\mathbf{x}')^2$$
$$= c^2(dt')^2 - c^2(dt')^2$$
because for light, $|d\mathbf{x}'| = c\,dt'$
$$= 0$$

Se we want to start with equations (1) -- (4), and then take the limit $d\tau \to 0$. But the limit is delicate because some quantities are singular.

## $\varphi$ and $\tau$

$\Sigma(r)\, dt/d\tau = \alpha \quad \rightarrow \infty \text{ as } d\tau \rightarrow 0 \qquad (1)$

$r^2\, d\varphi/d\tau = \lambda \quad \rightarrow \infty \text{ as } d\tau \rightarrow 0 \qquad (2)$

but

$$\frac{r^2\, d\varphi/d\tau}{\Sigma(r)\, dt/d\tau} = \frac{r^2\, d\varphi}{\Sigma(r)\, dt} = \frac{\lambda}{\alpha}$$

so $\lambda/\alpha$ remains finite as $d\tau \rightarrow 0$ .

Units: $\alpha$ is dimensionless; $\lambda \sim m^2/s$
Define $b = \lambda/(c\alpha)$ { units: m }.
Then the equations for $t(\varphi)$ is

$$\frac{dt}{d\varphi} = \frac{\alpha\, r^2}{\lambda \Sigma(r)} = \frac{r^2}{bc\, \Sigma(r)}$$

and that equation would be used to analyze the *time dependence.*

## $\varphi$ and r

$(dr/d\tau)^2 = [\, \alpha^2 - \Sigma(r)\, ]\, c^2 - \lambda^2\, \Sigma(r)/r^2 \qquad (4)$

$$\boxed{\frac{(dr/d\tau)^2}{(d\varphi/d\tau)^2} = \frac{|dr|^2}{|d\varphi|} = \frac{[\, \alpha^2 - \Sigma(r)\, ]\, c^2 - \lambda^2\, \Sigma(r)/r^2}{(\lambda/r^2)^2}}$$

$$(dr/d\varphi)^2 = r^4 \left\{ \frac{\alpha^2 c^2}{\lambda^2} - \frac{\Sigma\, c^2}{\lambda^2} - \frac{\Sigma}{r^2} \right\}$$

Now take the limit $d\tau \rightarrow 0$;

$\alpha^2 c^2/\lambda^2 = 1/b^2$ and $\lambda^2 \rightarrow \infty$;

thus $\qquad \Sigma = 1 - 2GM/c^2 r = 1 - r_S/r$

$$(dr/d\varphi)^2 = r^4/b^2 - r^2 + r_S\, r$$

where $r_S = 2GM/c^2$ .

That equation would be used to analyze the *trajectory, r(φ) or φ(r).*

The null geodesics have

$(dr/d\varphi)^2 = r^4/b^2 - r^2 + r_S\, r$ where $r_S = 2GM/c^2$ .

**Solution for M = 0; i.e., if there is no gravity, no curvature**

$$(dr/d\varphi)^2 = r^4/b^2 - r^2$$

Let $u = b/r$; or, $r = b/u$.

Then

$$\frac{dr}{d\phi} = \frac{-b}{u^2}\frac{du}{d\phi}$$

$$\frac{b^2}{u^4}\left(\frac{du}{d\phi}\right)^2 = \frac{b^2}{u^4} - \frac{b^2}{u^2}$$

$$\left(\frac{du}{d\phi}\right)^2 = 1 - u^2$$

$$u^2 + (du/d\phi)^2 = 1$$

The solution is $u = \sin\varphi$; that is,

$b = r\sin\varphi$.

The result is a straight line with $y = b$.

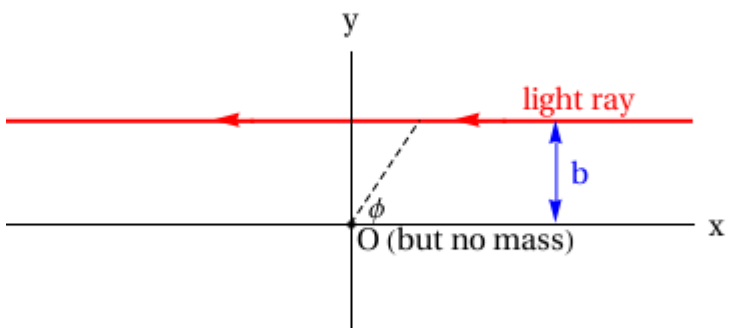
*When there is no gravity, no curvature, a light ray travels on a straight line.*

From the diagram, note that

b = the ***impact parameter.***

Also,

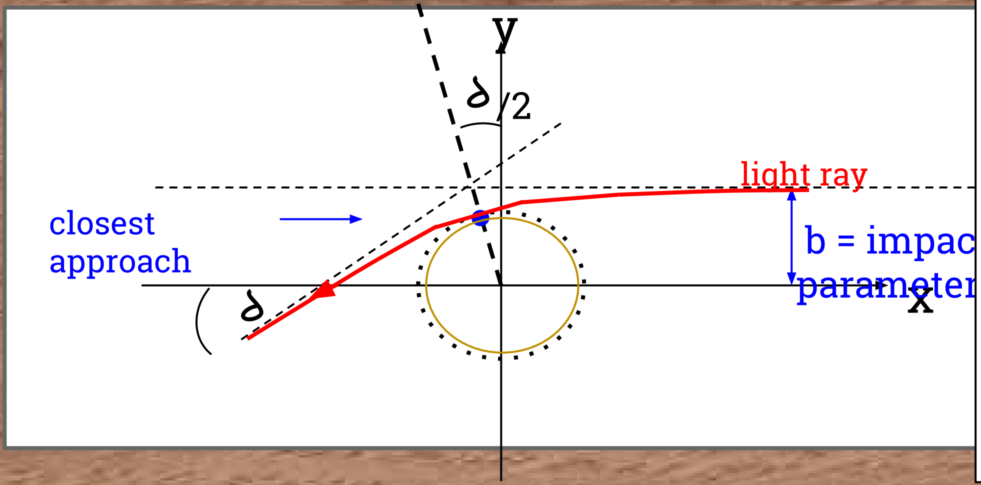
$r_c$ = the distance of closest approach to O = b.



# Deflection of light by the sun

*--- one of the three tests of general relativity proposed by Einstein.*

Light from a distant star approaches the sun, with impact parameter $b > R_{sol}$.
What will happen?




To solve:
$(dr/d\varphi)^2 = r^4/b^2 - r^2 + r_S\, r$ where $r_S = 2GM/c^2$

This term is very small. We could solve the equation approximately by perturbation theory; but we'll just solve it numerically with a computer program.

**Various parameters**

$b$ = impact parameter /input/
$r_c$ = distance of closest approach /calculate/
$\delta$ = the deflection angle /output/

Note from the diagram, the angle ($\varphi$) at $r = r_c$ is $\pi/2 + \delta/2$.

At closest approach, $dr/d\varphi = 0$;
so $r_c$ is the solution of this equation

$$r^4/b^2 - r^2 + r_s\, r = 0.$$

Because $r_s$ is small, $r_c \approx b - r_s/2$.
We'll need $r_c \leqq R_\odot$ ; $r_c = R_\odot$ is called "grazing incidence.

# SOLAR PARAMETERS

In[277]:=
```
Clear[Um, Us, Ukg]
Rsol = (6955 * 10 ^ - 3) * 10 ^ 8 * Um
G = (667 384 * 10 ^ - 5) * 10 ^ - 11 * Um ^ 3 * Us ^ - 2 * Ukg ^ - 1
c = (299 792 * 10 ^ - 5) * 10 ^ 8 * Um / Us
Ms = (19 891 * 10 ^ - 4) * 10 ^ 30 * Ukg
r0 = 2 * G * Ms / c ^ 2
Rsol / r0
```

Out[278]= 695 500 000 Um

Out[279]= $\frac{83\,423\ \mathrm{Um}^3}{1\,250\,000\,000\,000\,000\ \mathrm{Ukg}\ \mathrm{Us}^2}$

Out[280]= $\frac{299\,792\,000\ \mathrm{Um}}{\mathrm{Us}}$

Out[281]= 1 989 100 000 000 000 000 000 000 000 000 Ukg

Out[282]= $\frac{1\,037\,104\,308\,125\ \mathrm{Um}}{351\,075\,169}$

Out[283]= $\frac{390\,676\,448\,063\,200}{1\,659\,366\,893}$

# DEFLECTION OF A GRAZING LIGHT RAY

## CLOSEST APPROACH

In[366]:=
```
(* impact parameter for a grazing light ray *)
b = Rsol * (1 + 10 ^ - 5)
f[x_] := ((x * Um) ^ 4 / b ^ 2 - (x * Um) ^ 2 + r0 * x * Um) / Um ^ 2
fr = FindRoot[f[x], {x, b / Um},
  MaxIterations → 10, WorkingPrecision → 32]
rc = fr [[1]] [[2]]
rc * Um / Rsol
```

Out[366]= 695 506 955 Um

Out[368]= $\{x \to 6.9550547795504858999831875867787 \times 10^{8}\}$

Out[369]= $6.9550547795504858999831875867787 \times 10^{8}$

Out[370]= 1.0000078762833193242247573812766

## integral

In[377]:=
```
Um = 1
intgl = NIntegrate[1 / Sqrt[r ^ 4 / b ^ 2 - r ^ 2 + r0 * r], {r, rc, Infinity},
  AccuracyGoal → 16, PrecisionGoal → 16, WorkingPrecision → 24]
```

Out[377]= 1

Out[378]= 1.57080057419882043743099

## deflection $\delta$

In[379]:=
```
δ = -Pi + 2 * intgl
secondsofarc = δ * (360 / (2 * Pi)) * 3600
```

Out[379]= $8.49480784763639934 \times 10^{-6}$

Out[380]= 1.752179894799035525

For grazing incidence, the deflection of light by the sun is 1.75 seconds of arc.

Predicted by Einstein in 1917.

A famous expedition to observe stars behind the sun during a solar eclipse was supervised by Arthur Eddington, in 1919. He found that the observed positions of the stars were deflected toward the sun, just as predicted by Einstein's theory.

*Arthur Eddington's photograph of 29 May, 1919, taken of a full solar eclipse. The apparent displacement of a known star just behind the sun had been attributed by Newton to the bending of light rays by the sun. Einstein's prediction of a visual displacement twice that of Newton's was attributed by Einstein to the sun's warping of space. This photograph proved Einstein's prediction more accurate, and his theory correct:*