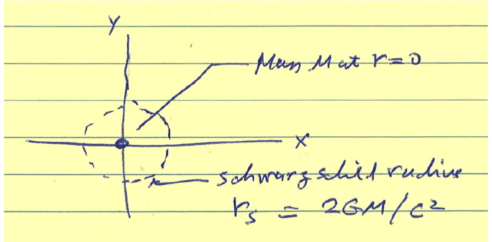
Light Rays and Black Holes

Black Holes

Q: What is a black hole?

A: An infinitely compressed mass

Picture



Q: Do they exist?

A: In theory, yes.

The invariant distance in coordinates (t, r, θ, φ) is ds:

$$(ds)^2 = \Sigma(r)\, c^2 (dt)^2 - (dr)^2/\Sigma(r) - r^2 (d\theta)^2 - r^2 \sin^2\theta\, (d\varphi)^2$$

$\Sigma(r) = 1 - 2GM/c^2 r = 1 - r_S/r$

$r_S = 2GM/c^2$

Q: Do they exist in reality?

A: Yes. There are several kinds.

Supermassive black holes

$$M \sim 10^5 \text{ -- } 10^9\, M_\odot$$

How did they form? ???

Are they common? Yes, many (or most) galaxies have a SMBH at the center.

What about the Milky Way galaxy?

Sagittarius A is a radio source at the center of the Milky Way galaxy, discovered in 1974.

Related: quasars, Seyfert galaxies, AGN

Collapsed stars

$$M \sim 3 \text{ -- } 30\, M_\odot$$

Gravitational collapse of a massive star ($M > \sim 10\, M_\odot$) produces a black hole.

Stellar mass black holes in the Milky Way galaxy: 15 candidates are known in X-ray binaries, with masses ranging from 4 to 14 $M_\odot$.

Primordial black holes

$M < \sim 10^9$ kg

People search for the Hawking radiation that they would produce, e.g., gamma-rays with photon energies $E > 10^3$ GeV.

Light rays (wave packets or photons) travel on null geodesics.
$(t, \ r, \theta, \varphi)$

Geodesic equations for a particle

$\Sigma(r)\, dt / d\tau = \alpha$ (1)

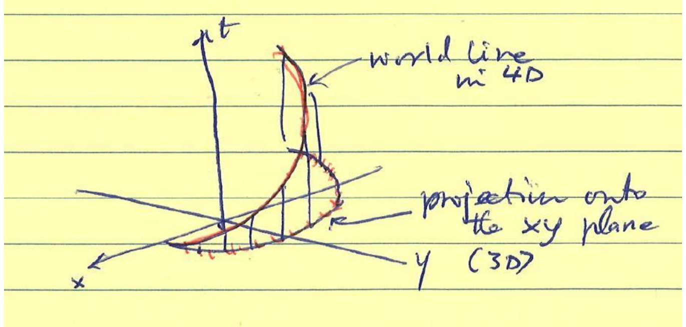
$r^2\, d\varphi / d\tau = \lambda$ (2)

$\theta = \pi / 2$ (3) (orbit in the xy plane)

$(dr / d\tau)^2 = [\alpha^2 - \Sigma(r)]\, c^2 - \lambda^2 \Sigma(r) / r^2$ (4)

where $d\tau$ = proper time.

$\Sigma(r) = 1 - r_S/r$



Equations for null geodesics
Take the limit $d\tau \to 0$, but carefully.

Let $\lambda / \alpha = bc$; b = impact parameter

$$\frac{dt}{d\phi} = \frac{dt/d\tau}{d\phi/d\tau} = \frac{\alpha}{\lambda}\frac{r^2}{\Sigma(r)} = \frac{r^2}{bc\,\Sigma(r)}$$

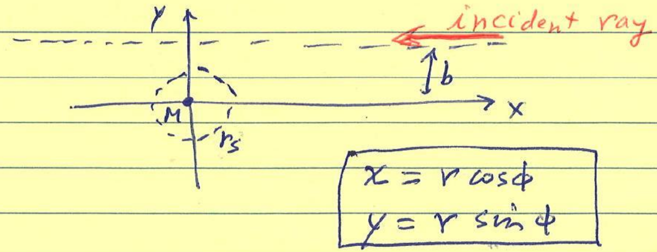
$$\left(\frac{dr}{d\phi}\right)^2 = \frac{(dr/d\tau)^2}{(d\phi/d\tau)^2} = \frac{[\alpha^2 - \Sigma(r)]c^2 - \lambda^2 \Sigma(r)/r^2}{(\lambda/r^2)^2}$$

$$= r^4\left\{\frac{1}{b^2} - \underbrace{0}_{\text{b/c } \lambda \to \infty} - (1 - r_s/r)/r^2\right\}$$

$$\left(\frac{dr}{d\phi}\right)^2 = r^4/b^2 - r^2 + r_s r$$

Trajectories of light rays
(spatial trajectory = the worldline projected onto the xy plane)

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{b^2} - r^2 + r_s r$$



Introduce scaled variables for numerical calculations

Let $b = r_S\,\beta$; i.e., $\beta = b/r_S$, a dimensionless constant.

Let $r = r_S\,\xi$; i.e., $\xi = r/r_S$, a dimensionless variable.

$$r_s^2\left(\frac{d\xi}{d\phi}\right)^2 = r_s^2\,\xi^4/\beta^2 - r_s^2\xi^2 + r_s^2\xi$$

$$\left(\frac{d\xi}{d\phi}\right)^2 = \xi^4/\beta^2 - \xi^2 + \xi$$

$$d\phi = \frac{\pm\, d\xi}{\sqrt{\xi^4/(\beta^2 - \xi^2 + \xi)}}$$

Integrate to get $\phi(\xi)$.

There are **unbound orbits** and **capture orbits**.

For **unbound orbits**, the light ray has a point of closest approach to the black hole.

At that point, $\xi = \xi_c$

where $\xi^4/\beta^2 - \xi^2 + \xi = 0$; and $\varphi = \varphi_c$

For $\varphi < \varphi_c$,

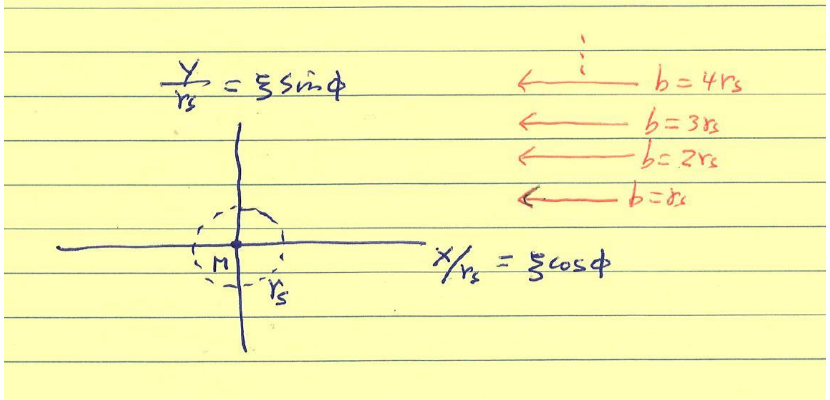
$$\varphi = -\int_{\xi}^{\infty} d\xi' / \mathrm{Sqrt}[\,\xi'^4/\beta^2 - \xi'^2 + \xi'\,]$$

For $\varphi > \varphi_c$,

$$\varphi = \int_{\xi_c}^{\xi} d\xi' / \mathrm{Sqrt}[\,\xi'^4/\beta^2 - \xi'^2 + \xi'\,]$$

For **capture orbits**, the light ray spirals into the origin, r = 0. There is no turning point.

$$\varphi = -\int_{\xi}^{\infty} d\xi' / \mathrm{Sqrt}[\,\xi'^4/\beta^2 - \xi'^2 + \xi'\,]$$



```
In[1057]:= Clear[ξ, ξmin];
           bs = {FontFamily → "Utopia", FontSize → 14};
           fancy = {BaseStyle → bs, AspectRatio → 13 / 14};
```

Light rays moving toward a black hole

```
In[1060]:= Jincoming[β_, ξ_] :=
             NIntegrate[1 / Sqrt[ξp^4 / β^2 - ξp^2 + ξp],
              {ξp, ξ, Infinity},
              WorkingPrecision → 32, AccuracyGoal → 20]
           Table[Jincoming[5, ξ], {ξ, {20, 15, 10, 5}}]
           Joutgoing[β_, ξc_, ξ_] :=
             NIntegrate[1 / Sqrt[ξp^4 / β^2 - ξp^2 + ξp],
              {ξp, ξc, ξ},
              WorkingPrecision → 32, AccuracyGoal → 20]

Out[1061]= {0.25257626352943676675738025455868, 0.33949121284087612513802101194706,
            0.52155473381178479398302226590165, 1.2564130786171602748721689440731}
```

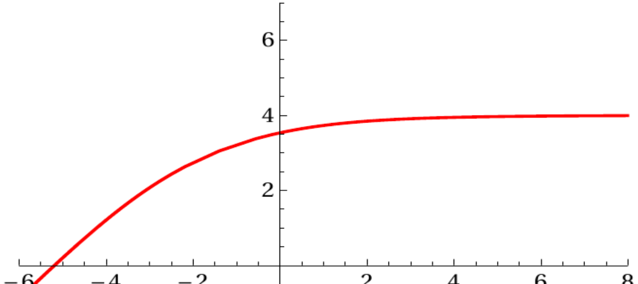
Integration for unbound orbits

```
In[1063]:= ib10 = 40; Nsteps = 100
           {β, ξ0} = {ib10 / 10, 10};
           FindRoot[ξ^4 / β^2 - ξ^2 + ξ, {ξ, β},
             WorkingPrecision → 32, AccuracyGoal → 20];
           ξmin = %[[1]][[2]]
           (**)
           dξ = (ξ0 - ξmin) / Nsteps;
           tbl = {};
           Do[ξ = ξ0 - dξ * j;
             ϕtemp = Jincoming[β, ξ];
             tbl = Join[tbl, {{ξ * Cos[ϕtemp], ξ * Sin[ϕtemp]}}],
             {j, 0, Nsteps - 1}];
           ϕc = Jincoming[β, ξmin];
           tbl = Join[tbl, {{ξmin * Cos[ϕc], ξmin * Sin[ϕc]}}];
           Do[ξ = ξmin + dξ * j;
             ϕtemp = ϕc + Joutgoing[β, ξmin, ξ];
             tbl = Join[tbl, {{ξ * Cos[ϕtemp], ξ * Sin[ϕtemp]}}],
             {j, 0, Nsteps - 1}];
           (**)
           lp[ib10] = ListPlot[tbl, fancy, Joined → True,
             PlotStyle → {Red, AbsoluteThickness[2]},
             PlotRange → {{-6, 8}, {-6, 7}}]

Out[1063]= 100

Out[1066]= 3.3502617411332921417792435963001

Out[1073]=
```

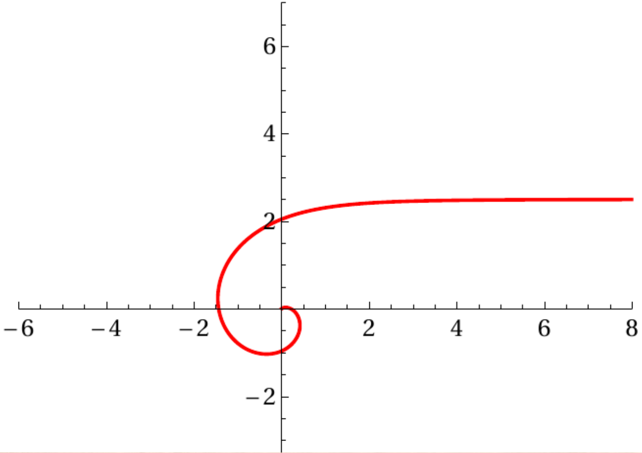


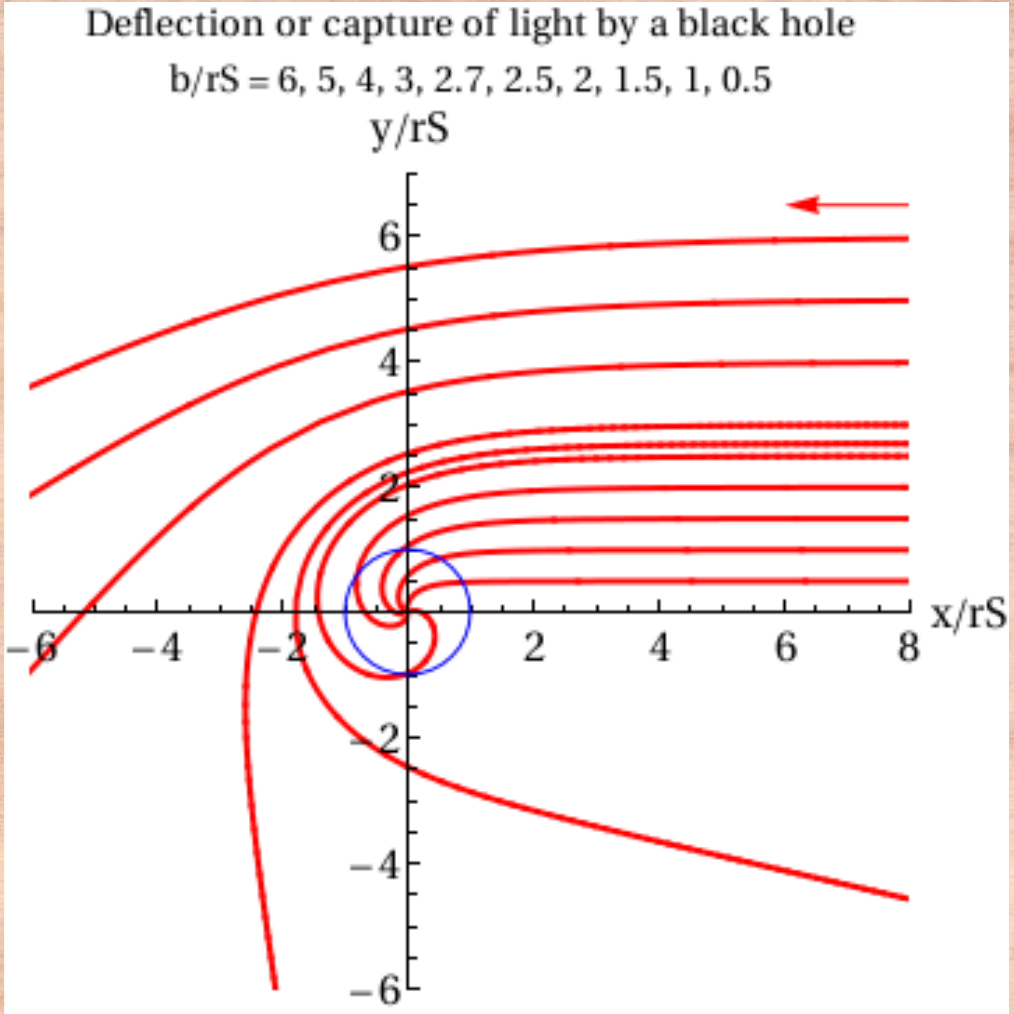
Integration for capture orbits

In[1074]:=
```
ib10 = 25; Nsteps = 1000;
{β, ξ0} = {ib10 / 10, 10};
ξmin = 0
(**)
dξ = (ξ0 - ξmin) / Nsteps;
tbl = {};
Do[ξ = ξ0 - dξ * j;
  ϕtemp = Jincoming[β, ξ];
  tbl = Join[tbl, {{ξ * Cos[ϕtemp], ξ * Sin[ϕtemp]}}],
  {j, 0, Nsteps - 1}];
(**)
lp[ib10] = ListPlot[tbl, fancy, Joined → True,
  PlotStyle → {Red, AbsoluteThickness[2]},
  PlotRange → {{-6, 8}, {-6, 7}}]
```

Out[1076]= 0

Out[1080]=




Deflection or capture of light by a black hole
b/rS = 6, 5, 4, 3, 2.7, 2.5, 2, 1.5, 1, 0.5
y/rS
x/rS
6
4
2
−2
−4
−6
−6
−4
−2
2
4
6
8