Lecture 1 — Review

$$\vec{F} = \frac{Q_1 Q_2}{4\pi\epsilon_0 r^2}\hat{r}$$

But this is not action at a distance.

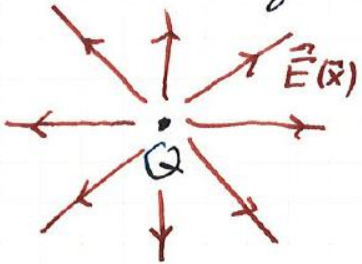
Every charge has an associated field, filling the space.

The field exerts forces on other charges.

Field Theory

$\vec{F} = q\,\vec{E}(\vec{x})$ ← the force on a test charge q at $\vec{x}$

Field theory



$$\vec{E}(\vec{x}) = \frac{Q\hat{r}}{4\pi\epsilon_0 r^2}$$

units of $\vec{E}$

N/C or V/m

Volt = $\frac{\text{joule}}{\text{coulomb}}$

$$\frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \ \frac{\mathrm{Nm}^2}{\mathrm{C}^2} \ \left(\text{or, } \frac{\mathrm{Vm}}{\mathrm{C}}\right)$$

The field equations of electrostatics

$$\nabla\cdot\vec{E} = \rho/\epsilon_0 \quad \text{and} \quad \nabla\times\vec{E} = 0.$$

<u>Divergence and Curl</u>

$$\vec{E}(\vec{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} = E_r\hat{r}$$

$$\nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = 0$$

But what about $\nabla\cdot\vec{E} = \rho/\epsilon_0$?

There is a singularity at $r = 0$.

We should have

$$\int_V \nabla\cdot\vec{E}\, d^3x = \oint_S \vec{E}\cdot d\vec{A} = \frac{Q}{4\pi\epsilon_0}\oint d\Omega$$

$$= Q/\epsilon_0$$

Gauss's law

$d\vec{A} = \hat{r}\, r^2 d\Omega$

Thus $\nabla\cdot\vec{E} = \frac{Q}{\epsilon_0}\delta^3(\vec{x})$

In general, then,

$$\nabla\cdot\vec{E} = \rho(\vec{x})/\epsilon_0 \quad \text{(superposition principle)}$$

<u>The curl</u>

In spherical coordinates,

$$\nabla\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin\theta\, F_\phi \end{vmatrix}$$

Now consider $\vec{F} = \hat{r}F_r(r)$

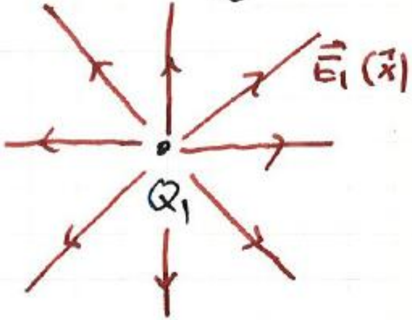
Then

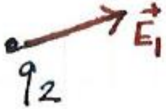
$$\nabla\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & 0 & 0 \end{vmatrix}$$

$$= 0$$

Electrostatics

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \times \vec{E} = 0$$





$$\vec{E}_1 = \frac{Q_1}{4\pi\epsilon_0 r^2}\hat{r}$$

$$\vec{F}_2 = q_2 \vec{E}_1 = \frac{Q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r}$$

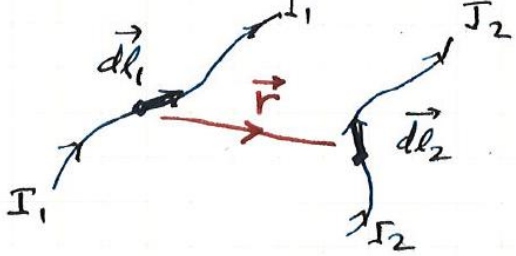
Field theory versus action at a distance

↳ Faraday; Maxwell

↓ obsolete after Faraday and Maxwell

Magnetostatics

↳ static magnetic systems



The force on $\vec{dl}_2$ due to the current in $\vec{dl}_1$ is

$$\delta\vec{F} = I_2\, \vec{dl}_2 \times d\vec{B}_1 \quad \text{(field theory)}$$

where

$$d\vec{B}_1 = \frac{\mu_0}{4\pi}\, \frac{I_1\, \vec{dl}_1 \times \hat{r}}{r^2} \quad \text{(field theory)}$$

So

$$\delta\vec{F} = \frac{\mu_0 I_1 I_2}{4\pi r^2}\, \vec{dl}_2 \times (\vec{dl}_1 \times \hat{r})$$

The force on the complete wire is

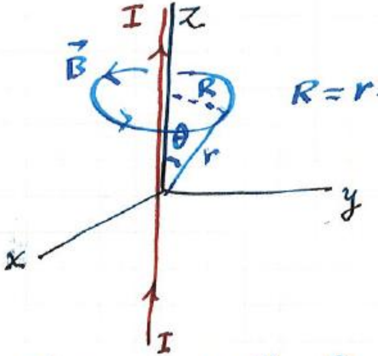
$$\vec{F} = \iint \frac{\mu_0 I_1 I_2}{4\pi r^2}\, \vec{dl}_2 \times (\vec{dl}_1 \times \hat{r})$$

This force is <u>not</u> <u>action at a distance</u>; it is a field effect.

$$\mu_0 = 4\pi \times 10^{-7}\ \frac{\mathrm{Tm}}{\mathrm{A}}$$

Example

a long straight wire carrying current I



$R = r\sin\theta$

$$\vec{B}(\vec{x}) = \frac{\mu_0 I}{2\pi R}\,\hat{\phi}$$

$$\nabla\times\vec{B} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & 0 & r\sin\theta\,\dfrac{\mu_0 I}{2\pi r\sin\theta} \end{vmatrix}$$

$$\nabla\times\vec{B} = 0$$

Why not $\nabla\times\vec{B} = \mu_0\vec{J}$?

Again, there is a singularity, at $R = 0$.

We should have

Stokes thm

$d\vec{l} = \hat{\phi}\,dl$

$$\int_{\text{Disk}} (\nabla\times\vec{B})\cdot d\vec{A} = \oint_{\text{circle}} \vec{B}\cdot d\vec{l}$$

$$= \frac{\mu_0 I}{2\pi R}\,2\pi R = \mu_0 I$$

Thus $\nabla\times\vec{B} = \mu_0 I\,\delta(x)\,\delta(y)\,\hat{k}$

Or, in general,

$$\nabla\times\vec{B} = \mu_0 \underbrace{\vec{J}(\vec{x})}_{\text{current density}}$$

(current per unit area)

Check units:

$$\frac{1}{m}T = \frac{Tm}{A}\,\frac{A}{m^2} \checkmark$$

Exercise: Prove $\nabla\cdot\vec{B} = 0$.

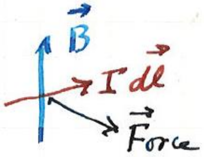
Magnetostatics

- The field equations of magnetostatics are

$$\nabla \times \vec{B} = \mu_0 \vec{J} \quad \text{and} \quad \nabla \cdot \vec{B} = 0$$

- The force on a current element is

$$d\vec{F} = I \, d\vec{l} \times \vec{B}$$



- Units $N = A\,m\,T$

$$T = \frac{N}{Am} = \frac{J}{Am^2} = \frac{VC}{C/s\; m^2} = \frac{Vs}{m^2}$$

$$1\,T = 1 \; \frac{Vs}{m^2}$$

- The force on a charged particle is

$$\vec{F} = q\,\vec{v} \times \vec{B}$$

$\left(\text{Units: } C \frac{m}{s} T = A\,m\,T = N \checkmark\right)$

END of the REVIEW