Maxwell's Equations (Chapter 11)

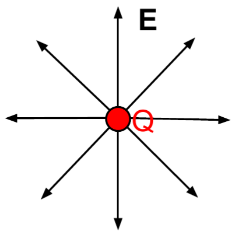
The fundamental equations for electric and magnetic fields.

Let $\rho(\mathbf{x},t)$ = charge density

and $\mathbf{J}(\mathbf{x},t)$ = current density

(1) Gauss's law $\quad \nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \qquad (1)$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0\, r^2}\,\hat{\mathbf{r}}$$



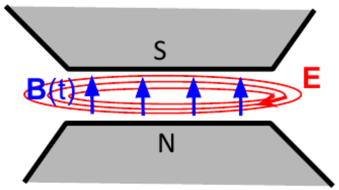
(2) Gauss's law, $\quad \nabla \cdot \mathbf{B} = 0 \qquad (2)$

*but magnetic monopoles do not exist**

Or, the equivalent integral equations

(3) Faraday's law $\quad \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \qquad (3)$

E** curls around the change of **B



(4) The Ampère-Maxwell law

$$\nabla \times \mathbf{B} = \mu_0\, \mathbf{J} + \text{“}X\text{”}$$

Maxwell realized that "X" cannot be zero:

$\nabla \cdot (\nabla \times \mathbf{B})$ is always zero --- that's a mathematical identity; but

$\nabla \cdot \mathbf{J}$ is not always zero.

Conservation of charge implies

$$\nabla \cdot \mathbf{J} = -\,\partial \rho/\partial t \qquad \text{(“continuity equation”)}$$

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \text{“}X\text{”}$

$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \mu_0 \nabla \cdot \mathbf{J} + \nabla \cdot \text{“}X\text{”}$

$\nabla \cdot \mathbf{J} = -\partial\rho/\partial t = \nabla \cdot (-\varepsilon_0 \, \partial \mathbf{E}/\partial t)$

$0 = \nabla \cdot (-\mu_0\varepsilon_0 \, \partial \mathbf{E}/\partial t) + \nabla \cdot \text{“}X\text{”}$

Simplest possibility is $\text{“}X\text{”} = \mu_0\varepsilon_0 \, \partial \mathbf{E}/\partial t$.

(4) The Ampere-Maxwell law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\varepsilon_0 \, \partial \mathbf{E}/\partial t \qquad (4)$$

Equations (1) -- (4) were published by Maxwell in 1864. We still use the same equations today.

Maxwell's *displacement current*

Define $\mathbf{J}_D = \varepsilon_0 \, \partial \mathbf{E}/\partial t$

Then $\nabla \times \mathbf{B} = \mu_0 \, (\mathbf{J} + \mathbf{J}_D)$

So a there are two different ways that a magnetic field can be created:
from the charge current **J** *(carried by particles) or from the displacement current* $\mathbf{J}_D$ *(a field effect).*
Note the similarity to Faraday's law!

Fields without charges

Consider $\rho = 0$ and $\mathbf{J} = 0$.

Now the field equations are...

$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \, \partial \mathbf{E} / \partial t$$

This effect was not observed experimentally in the time of Maxwell.

These "free field equations" can be satisfied by waves of electric and magnetic fields ⟹ electromagnetic waves.

Wave solutions

Try these functions:

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 \sin(\, \mathbf{k} \cdot \mathbf{x} - \omega t \,)$$

$$\mathbf{B}(\mathbf{x},t) = \mathbf{B}_0 \sin(\, \mathbf{k} \cdot \mathbf{x} - \omega t \,)$$

(1) $\nabla \cdot \mathbf{E} = \mathbf{k} \cdot \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) = 0$

Therefore $\mathbf{k} \cdot \mathbf{E}_0 = 0$; the electric field oscillations are transverse, i.e., perpendicular to the direction of wave motion ($\mathbf{k}$).

(2) $\nabla \cdot \mathbf{B} = \mathbf{k} \cdot \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) = 0$

Therefore $\mathbf{k} \cdot \mathbf{B}_0 = 0$; the magnetic field oscillations are transverse, i.e., perpendicular to the direction of wave motion ($\mathbf{k}$).

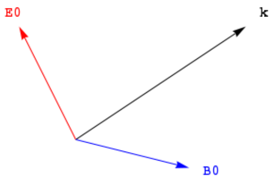
(3) $\nabla \times \mathbf{E} + \partial \mathbf{B} / \partial t =$

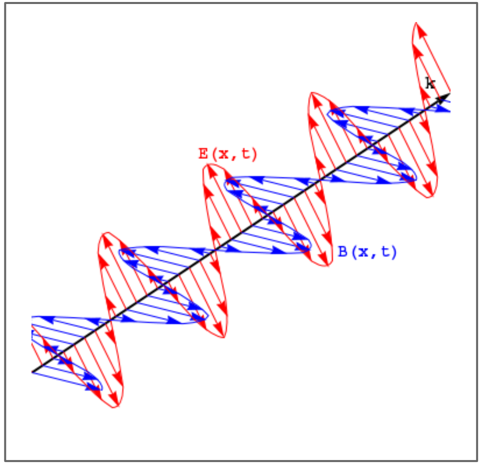
$$(\, \mathbf{k} \times \mathbf{E}_0 - \omega \, \mathbf{B}_0 \,) \, \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) = 0$$

Therefore $\mathbf{B}_0 = (\mathbf{k} \times \mathbf{E}_0)/\omega$; $\mathbf{B}_0$ is perpendicular to $\mathbf{E}_0$.

Wave solutions

- $\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 \sin(\mathbf{k}\cdot\mathbf{x} - \omega t)$
- $\mathbf{B}(\mathbf{x},t) = \mathbf{B}_0 \sin(\mathbf{k}\cdot\mathbf{x} - \omega t)$
- $\mathbf{k}$, $\mathbf{E}_0$ and $\mathbf{B}_0$ form an orthogonal triad of vectors
- $\mathbf{B}_0 = (\mathbf{k} \times \mathbf{E}_0)/\omega$





k : the wave vector; direction of propagation; magnitude =2 pi /lambda

ω : the angular frequency = 2 pi / T .

(4) The Ampere-Maxwell law

$$\nabla \times \mathbf{B} = \mu_0\varepsilon_0\, \partial\mathbf{E}/\partial t$$

$$\mathbf{k} \times \mathbf{B}_0 = \mu_0\varepsilon_0\ (-\omega\, \mathbf{E}_0)$$

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0)/\omega = -k^2\, \mathbf{E}_0/\omega$$

And so… $k^2 = \omega^2 \mu_0\varepsilon_0$

the final requirement for the four Maxwell equations.

The wave velocity

"Phase Velocity" of a harmonic wave = wavelength/period of oscillation =

$(2\ \text{pi}/k) / (2\ \text{pi}/\omega) = \omega/k = 1/(\mu_0\varepsilon_0)^{1/2}$

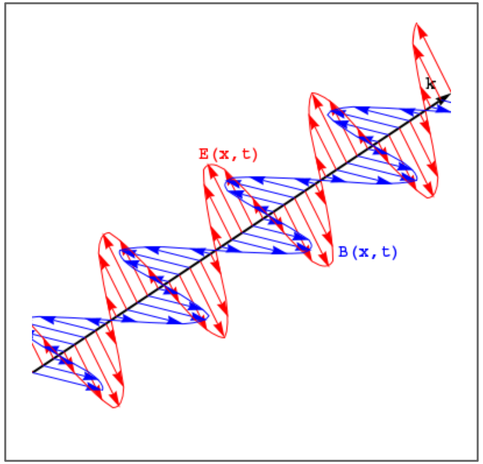
$= 3.00 \times 10^8$ m/s ,

and that is the speed of light!

Result

Phase velocity = $\omega/k = 1/(\mu_0\varepsilon_0)^{1/2} = c$, the speed of light.

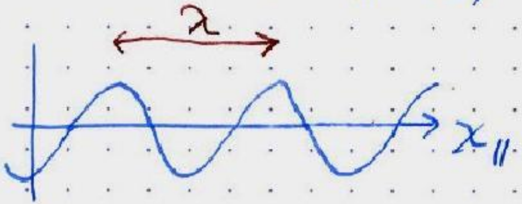
So Maxwell concluded that *light is an electromagnetic wave.*



Furthermore, nothing in the theory limits the wavelength. So the theory implies the existence of an infinite spectrum of EM waves.

$\sin(\vec{k}\cdot\vec{x} - \omega t)$

$\hookrightarrow = \sin(k x_{\parallel} - \omega t)$



wavelength $\lambda = \frac{2\pi}{k}$

period $T = \frac{2\pi}{\omega}$

frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$

phase velocity $v_{ph.} = \frac{\lambda}{T} = \lambda f$

$v_{ph.} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$