Maxwell's Equations - Energy and Potentials

The fundamental equations, for isolated charges and currents in empty space,

$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$ and $\nabla \cdot \mathbf{B} = 0$

$\nabla \times \mathbf{E} = - \partial B/\partial t$

and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \, \partial \mathbf{E}/\partial t$

Energy and momentum of E and B fields

We already know…

== the electric field energy density

$u_E(\mathbf{x},t) = ½ \, \varepsilon_0 \, E^2$

[Units: $(C^2/Nm^2)\,(N/C)^2 = N/m^2 = J/m^3$]

== the magnetic field energy density

$u_M(\mathbf{x},t) = B^2/(2\mu_0)$

[Units: $T^2\,(A/Tm) = (Vs/m^2)\,(C/sm) = J/m^3$]

The Energy Flux

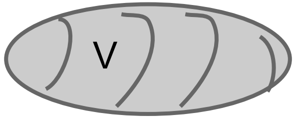
= energy flow per unit area per unit time

Define **S(x**,t**)** = energy flux

Define $\mathbf{S}(\mathbf{x},t)$ = energy flux

Consider a volume V. Let U be the total field energy inside the volume V.

By conservation of energy,



$$dU/dt = -\oint \mathbf{S}\cdot d\mathbf{A} \quad - dW/dt$$

(- energy flowing out through the surface)

(- work done on particles)

$$dU/dt = \int_V \left(\partial u_E/\partial t + \partial u_B/\partial t \right) d^3x$$

Derivation of Poynting's vector $\mathbf{S}(\mathbf{x},t)$

$$\partial u_E/\partial t = \varepsilon_0 \, \mathbf{E}\cdot \partial \mathbf{E}/\partial t$$
$$= \varepsilon_0 \, \mathbf{E}\cdot (\nabla\times\mathbf{B} - \mu_0 \mathbf{J})/\mu_0\varepsilon_0$$

by the Ampere-Maxwell law

$$= (1/\mu_0)\, \mathbf{E}\cdot(\nabla\times\mathbf{B}) - \mathbf{J}\cdot\mathbf{E}$$

$$\partial u_B/\partial t = (1/\mu_0)\, \mathbf{B}\cdot \partial\mathbf{B}/\partial t$$
$$= -(1/\mu_0)\, \mathbf{B}\cdot(\nabla\times\mathbf{E})$$

by Faraday's law

Note: $\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \varepsilon_{ijk}\, \partial_i (E_j B_k)$

$$= \varepsilon_{ijk} \{ (\partial_i E_j) B_k + E_j (\partial_i B_k) \}$$
$$= (\nabla\times\mathbf{E})\cdot\mathbf{B} - \mathbf{E}\cdot(\nabla\times\mathbf{B})$$

Thus,

$$\partial u/\partial t = -\nabla\cdot(\mathbf{E}\times\mathbf{B})/\mu_0 - \mathbf{J}\cdot\mathbf{E}$$

$$dU/dt = -\oint(\mathbf{E}\times\mathbf{B})/\mu_0\cdot d\mathbf{A} \quad - dW/dt$$

Check the work done on the charged particles in an infinitesimal volume dV:

$$dW/dt = \mathbf{F}\cdot\mathbf{dx}/dt = dQ\,\mathbf{E}\cdot\mathbf{dx}/dt$$
$$= \mathbf{E}\cdot(\mathbf{J}\,dA\,dt)\,dx/dt$$
$$= \mathbf{E}\cdot\mathbf{J}\;dV$$

Thus the energy flux is equal to the Poynting vector, $\mathbf{S(x,t)} = (\mathbf{E}\times\mathbf{B})/\mu_0$.

Example: Energy in a plane electromagnetic wave

$$\mathbf{E} = \boldsymbol{i}\,E_0\sin(kz-\omega t)$$
$$\mathbf{B} = \boldsymbol{j}\,B_0\sin(kz-\omega t)$$

Energy densities

$$u_E = (\varepsilon_0/2)\,E_0^2\sin^2(kz-\omega t)$$
$$u_B = (1/2\mu_0)\,B_0^2\sin^2(kz-\omega t)$$
$$= (\varepsilon_0/2)\,E_0^2\sin^2(kz-\omega t)$$

(because $B_0 = E_0/c$ and $\mu_0\varepsilon_0 = 1/c^2$)

The energy flux is

$$\mathbf{S}(x,t) = (\mathbf{E}\times\mathbf{B})/\mu_0$$
$$= \boldsymbol{k}\,E_0^2/(\mu_0 c)\,\sin^2(kz-\omega t)$$
$$= \boldsymbol{k}\,u\,c$$

Potential Functions

$\mathbf{A}(\mathbf{x},t)$ and $V(\mathbf{x},t)$

(vector potential and scalar potential)

$\nabla \cdot \mathbf{B} = 0$ implies that we can write $\mathbf{B} = \nabla \times \mathbf{A}$

Now,

$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t = -\nabla \times (\partial \mathbf{A}/\partial t)$

(Faraday's law)

$\nabla \times (\mathbf{E} + \partial \mathbf{A}/\partial t) = 0$;

that means we can write

$\mathbf{E} + \partial \mathbf{A}/\partial t = -\nabla V$

So, $\mathbf{B} = -\nabla \times \mathbf{A}$

and $\mathbf{E} = -\nabla V - \partial \mathbf{A}/\partial t$

Gauge transformations and gauge invariance

The functions **A** and V are not uniquely determined by **E** and **B**

Let $\lambda(x,t)$ be an arbitrary scalar function.

Define the "gauge transformation"

$\mathbf{A}' = \mathbf{A} + \nabla\lambda$ and $V' = V - \partial\lambda/\partial t$

Then $\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times (\nabla\lambda) = \nabla \times \mathbf{A} = \mathbf{B}$

and

$-\nabla V' - \partial \mathbf{A}'/\partial t = -\nabla V - \partial \mathbf{A}/\partial t$

$+ \nabla(\partial\lambda/\partial t) - \partial(\nabla\lambda)/\partial t$

$= \mathbf{E}$

I.e., $\{\mathbf{A}', V'\}$ are equivalent potentials (though not equal) to $\{\mathbf{A}, V\}$.

Homework Exercise 11.5:

In the Lorentz gauge (a certain choice of **A** and V) the potentials **A(x,t)** and V(x,t) obey the wave equation.