Electric and magnetic units …

$F = q\,(E + v \times B)$

$[E] \;=\; m/s\,[B]$

$V/m \;=\; Tm/s$

Electromagnetic Waves in Vacuum

Maxwell's Equations in empty space,

$\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$

$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$; $\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \, \partial \mathbf{E} / \partial t$

We'll construct the general *polarized plane wave solution*

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 \exp[\, i\, (\mathbf{k} \cdot \mathbf{x} - \omega\, t)]$$

This is an idealized mathematical solution; the wave fronts are infinite planes perpendicular to **k** ; and perfectly polarized (i.e., the direction of oscillation is constant).

Important: the polarized plane waves are a *complete set* of solutions.

Complex exponentials

$\exp(i\,\theta) = \cos(\theta) + i \sin(\theta)$ (Euler)

or

$A \exp(i\,\theta) = |A| \{\cos(\theta + d) + i \sin(\theta + d)\}$

where

$A = |A| \exp(id)$.

Of course **E** must be a real function. But is is more convenient to solve the equations using complex exponential functions. Just remember, the physical solution is the **Real Part** of the complex solution.

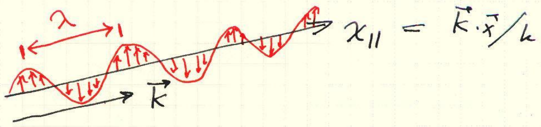
At the end of the calculation we take the real part to get the physical solution.

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{x} - \omega\, t)$$

Polarized plane wave

$$\vec{E}(\vec{x},t) = \vec{E}_0 \; e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

- Harmonic time dependence; period $T = \frac{2\pi}{\omega}$
- Propagation in the direction of $\vec{k}$.



- Wavelength $\lambda = \frac{2\pi}{k}$

$$e^{ik(x_{\parallel}+\lambda)} = e^{ikx_{\parallel}} \underbrace{e^{2\pi i}}_{1} = e^{ikx_{\parallel}}$$

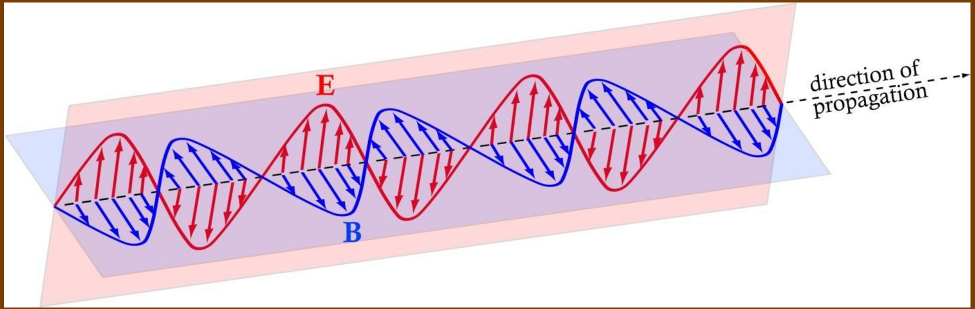
- Phase velocity

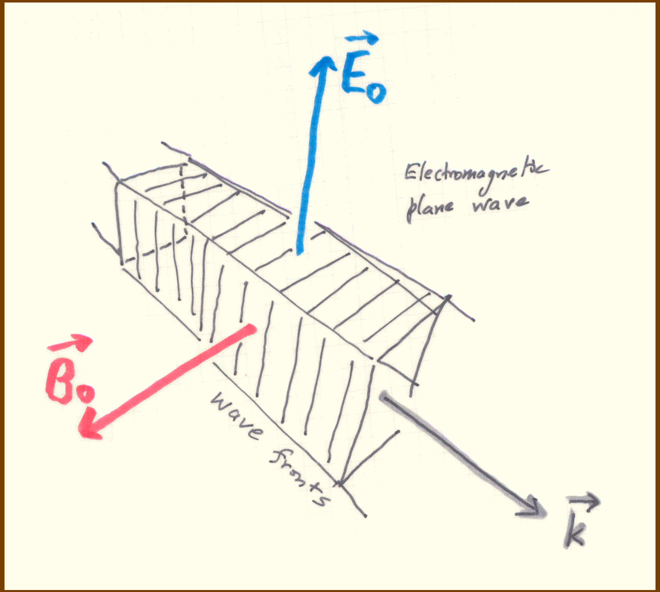
$$\vec{k}\cdot\vec{x} - \omega t = \text{constant}$$

$$k x_{\parallel} - \omega t = \text{constant}$$

$$k \, \Delta x_{\parallel} - \omega \, \Delta t = 0$$

$$v_{phase} = \frac{\Delta x_{\parallel}}{\Delta t} = \frac{\omega}{k}$$





<u>Maxwell's equations must be satisfied</u>

$\mathbf{E}(\mathbf{x},t) = \mathbf{E_0} \exp[\, i\, (\mathbf{k}\cdot\mathbf{x} - \omega\, t)]$

(1) $\nabla \cdot \mathbf{E} = 0$

$$i\mathbf{k}\cdot\mathbf{E_0} \exp[\, i\, (\mathbf{k}\cdot\mathbf{x} - \omega\, t)] = 0$$

$$\mathbf{k}\cdot\mathbf{E_0} = \mathbf{0}\ ;$$

The electric field oscillates in a direction perpendicular to the direction of propagation; electromagnetic waves are *transverse waves*; the *polarization direction*.

(3) $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$

$$i\mathbf{k} \times \mathbf{E_0} \exp[\, i\, (\mathbf{k}\cdot\mathbf{x} - \omega\, t)] = -\partial \mathbf{B} / \partial t$$

$$\mathbf{B} = (\mathbf{k} \times \mathbf{E_0}\ /\omega\ ) \exp[\, i\, (\mathbf{k}\cdot\mathbf{x} - \omega\, t)]$$

Or,

$$\mathbf{B} = \mathbf{B_0} \exp[\, i\, (\mathbf{k}\cdot\mathbf{x} - \omega\, t)] \text{ and } \mathbf{B}_0 = \mathbf{k} \times \mathbf{E_0}\ /\omega$$

Note that **k, $\mathbf{E_0}$** and **$\mathbf{B_0}$** form an orthogonal triad of vectors. Also, $\mathbf{E_0} \times \mathbf{B_0} = \mathbf{k}\ E_0^{\,2} / \omega$

(3) $\nabla \cdot \mathbf{B} = 0$

This equation is already satisfied because

$\nabla \cdot \mathbf{B} = i\mathbf{k}\cdot \mathbf{B_0} \exp[\, i\, (\mathbf{k}\cdot\mathbf{x} - \omega\, t)]$

and $\mathbf{k}$ and $\mathbf{B_0}$ are perpendicular.

(4) $\nabla \times \mathbf{B} = \mu_0\varepsilon_0\, \partial\mathbf{E}/\partial t$

$$i\, \mathbf{k} \times \mathbf{B_0} = \mu_0\varepsilon_0\, (-i\, \omega\, \mathbf{E_0})$$

$$\mathbf{k} \times \mathbf{B_0} = \mathbf{k} \times (\mathbf{k} \times \mathbf{E_0})\, /\omega = -\, k^2\, \mathbf{E_0}\, /\omega$$

Thus, $\quad k^2 /\omega = \mu_0\varepsilon_0\, \omega$

$$\boxed{\omega = c\, k \quad \text{where} \quad c = 1/(\mu_0\varepsilon_0)^{1/2}}$$

Exercise.

Show that $B_0 = E_0 /c$.

[Units: T = (V/m) /(m/s) = Vs /m^2]

Electromagnetic waves in vacuum

$v_{phase} = \omega / k = 1 / (\mu_0 \varepsilon_0)^{1/2} = c$

$v_{group} = d\omega / dk = 1/(\mu_0 \varepsilon_0)^{1/2} = c$

$c = 1/(\mu_0 \varepsilon_0)^{1/2} = 3.00 \times 10^8$ m/s

- All wavelengths have the same speed.
- The wave speed does not depend on the frame of reference (A. Einstein). This assumes that Maxwell's equations have the same form in all inertial frames (theory of relativity).
- $B_0 = E_0/c$ independent of wavelength

The solution constructed here is called the *polarized plane wave solution*.

It is an idealized e.m. wave: perfectly polarized and coherent.

The wave fronts are infinite planes perpendicular to k ("coherence").

The electric field oscillates only in one direction ("polarized"); the magnetic field oscillates in a perpendicular direction.

The polarized plane wave solution

- **E** x **B** is everywhere in the direction of **k**.
  (= direction of flow of energy)
- On any plane perpendicular to **k**, **E**(**x**,t) and **B**(**x**,t) are *independent of **x*** ("coherence") (and oscillating in t).

Completeness

The ideal polarized plane waves are important because the are a *complete set of solutions* to Maxwell's equations (in vacuum).

Superposition principle

Any solution can be written as a superposition of plane waves.

Example. A finite pulse of light (a "flash") would be an outgoing spherical wave with radius r = c t and finite thickness δr.
That can be written as a superposition of plane waves.

The fact that any solution of Maxwell's equations can be written as a superposition of plane waves is an example of ***Fourier's Theorem.***

<u>Polarized plane waves</u>

E x B is everywhere in the direction of

That can be written as a superposition of plane waves.