Plane waves and wave packets

Consider harmonic waves for which the phase velocity is constant: $v_{phase} = \omega / k = c$.

★ Plane wave

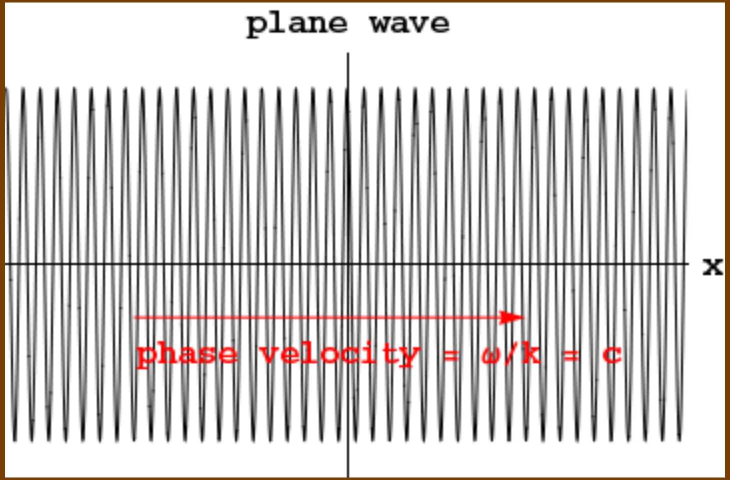
$= \cos(kx - \omega t)$; or $\exp[i(kx - \omega t)]$ {Re implied}

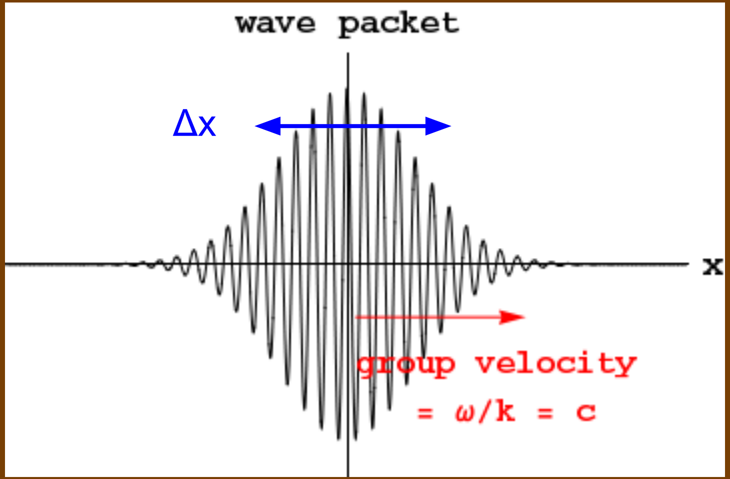
The wave extends to infinite distance along the x axis.

★ Wave packet

The wave has finite extent;
the spatial width of the wave packet = Δx.

Theorem. Any wave packet that satisfies the wave equation can be written as a superposition of plane waves. (quite remarkable!)





The 1D wave equation

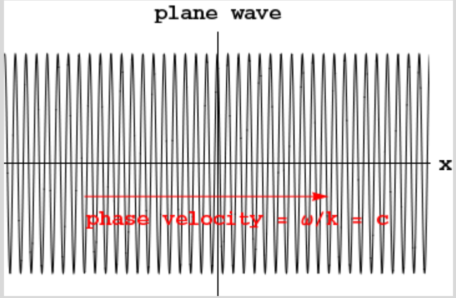
Harmonic wave $\psi(x,t) = A e^{i(kx-\omega t)}$

$$\frac{\partial \psi}{\partial t} = -i\omega\psi \quad \text{and} \quad \frac{\partial^2 \psi}{\partial t^2} = -\omega^2\psi$$

$$\frac{\partial \psi}{\partial x} = ik\psi \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} = -k^2\psi$$

We'll assume that the phase velocity is constant : $\omega/k = c$

So $\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0$ (wave equation)



The 1D wave equation

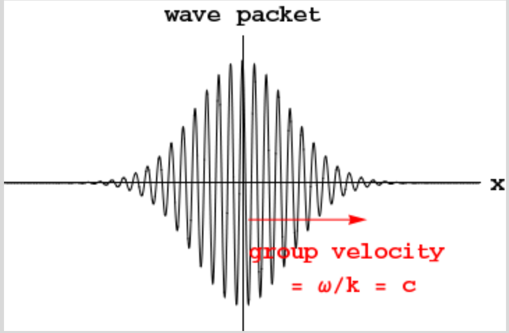
Now consider $g(x,t) = G(x-ct)$

$$\frac{\partial g}{\partial x} = G' \quad \text{and} \quad \frac{\partial g}{\partial t} = -cG'$$

$$\frac{\partial^2 g}{\partial t^2} - c^2 \frac{\partial^2 g}{\partial x^2} = c^2 G'' - c^2 G'' = 0$$

So this is a solution of the wave equation for any function $G(\xi)$ [$\xi = x - ct$]

[The harmonic wave $= A e^{ik\xi}$.]



Now write g(x,t) as a superposition of harmonic waves

$$g(x,t) = \int_{-\infty}^{\infty} \hat{G}(k)\, e^{i(kx-\omega t)}\, \frac{dk}{2\pi}$$

or

$$G(\xi) = \int_{-\infty}^{\infty} \hat{G}(k)\, e^{ik\xi}\, \frac{dk}{2\pi}$$

$\xi = x - ct$

$k\xi = k\,(x - ct) = kx - \omega t$

Mathematics of Fourier Transforms

$$\hat{G}(k) = \int_{-\infty}^{\infty} G(\xi)\, e^{-ik\xi}\, d\xi$$

Example : The Gaussian wave packet

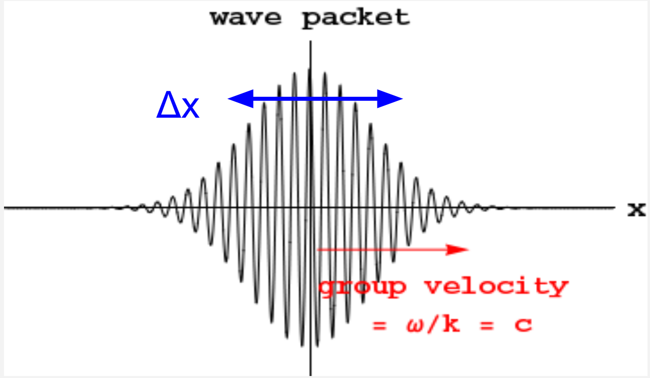
The Gaussian wave packet

$$g(x,t) = A\, e^{i(k_0 x - \omega_0 t)}\, e^{-(x-ct)^2/2a^2}$$

i.e.,

$$G(\xi) = A\, e^{ik_0\xi}\, e^{-\xi^2/2a^2}$$

What is $\hat{G}(k)$?



$\Delta x \sim 2a$

Gaussian integrals

(1) Consider $I(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$

$$I^2(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} \int_{-\infty}^{\infty} e^{-\alpha y^2} \, dx \, dy$$

$$= \int_{0}^{\infty} e^{-\alpha r^2} \, 2\pi r \, dr$$

$$= \pi \int_{0}^{\infty} e^{-\alpha s} \, ds = \pi/\alpha$$

(2) Consider $I_2(\alpha,\beta) = \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{\beta x} dx = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/4\alpha}$

$$= \int_{-\infty}^{\infty} e^{-\alpha\left[(x-\beta/2\alpha)^2 - \beta^2/4\alpha^2\right]} dx$$

$$= \underbrace{\int_{-\infty}^{\infty} e^{-\alpha x'^2} dx'}_{\sqrt{\pi/\alpha}} \; e^{+\beta^2/4\alpha} = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/4\alpha}$$

__Example__.
The Gaussian wave packet ($\xi = x - ct$)

$$\hat{G}(k) = \int_{-\infty}^{+\infty} G(\xi) \exp[-ik\xi] \, d\xi$$

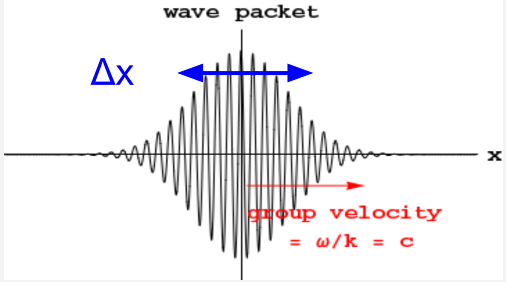
(3) $\hat{G}(k) = \int_{-\infty}^{\infty} A e^{ik_0\xi} e^{-\xi^2/2a^2} e^{-ik\xi} d\xi$

$$= \int_{-\infty}^{\infty} A\, e^{i(k_0-k)\xi} e^{-\xi^2/2a^2} d\xi$$

$$= A\, I_2\left(\frac{1}{2a^2},\, i(k_0-k)\right)$$

$$= A \sqrt{\frac{\pi}{(1/2a^2)}}\; e^{\frac{1}{4}(2a^2)(i(k_0-k))^2}$$

$$= A \sqrt{2\pi}\; a\; e^{-\frac{1}{2}(k-k_0)^2 a^2}$$



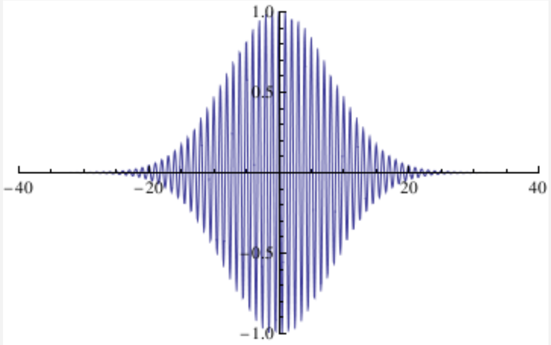
Result

The Gaussian wave packet

$$g(x,t) = A \exp[\, i(k_0 x - \omega_0 t)]\, \exp[-(x-ct)^2/2a^2\,]$$

$$= \int_{-\infty}^{+\infty} \hat{G}(k)\, \exp[ik(x-ct)]\, dk/2\pi$$

where $$\hat{G}(k) = A\sqrt{2\pi}\, a\, e^{-\frac{1}{2}(k-k_0)^2 a^2}$$



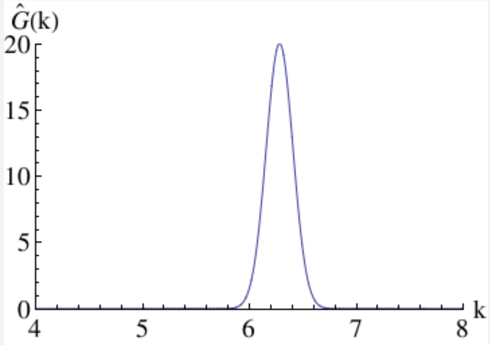


Figure shows case w/

l_0 = 1, a = 8...

x-width : $\Delta x \sim 2a$

k-width : $\Delta k \sim 2/a$

$\Delta x\, \Delta k \sim 1$

Numerical Studies

The Gaussian wave packet is

$$g(x,t) = A \exp[\, i(k_0x-\omega_0t)] \exp[-(x-ct)^2/2a^2\,]$$

Pick some interesting parameter values...

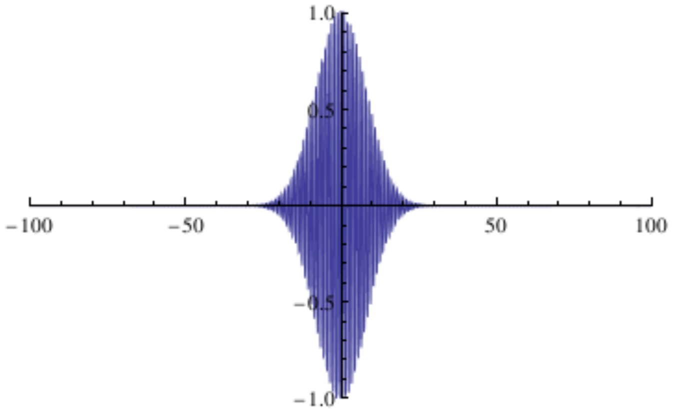
wavelength = 1 unit

spatial width ~ 2 a = 16 u ; a = 8 u.

$k_0 = 2\pi/\text{wavelength} = 6.28$ /u.

amplitude A = 1

A graph of the wave packet:

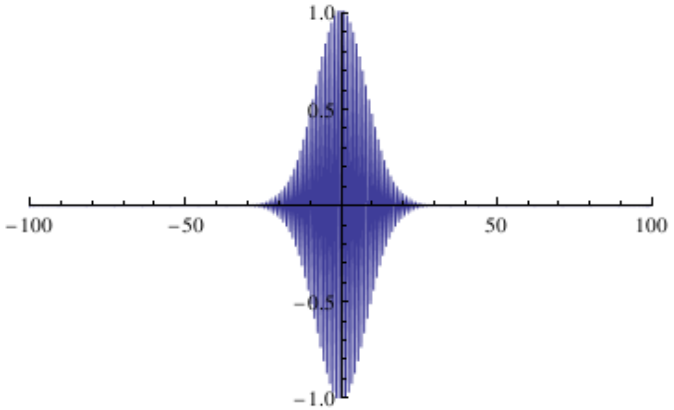


Now compute the superposition of plane waves with a computer program, and compare to the wave packet. Analytically they are the same; numerically they can only be approximately equal, but if we use a good enough approximation they should be almost equal.

Approximate

$$\int_{-\infty}^{+\infty} dk\,(\ldots) \approx \sum \varepsilon\,(\ldots) \quad \text{where } k = n\varepsilon$$

Result with ε = 0.05; approximating the integral by the sum of 960 k values:



Numerical Studies

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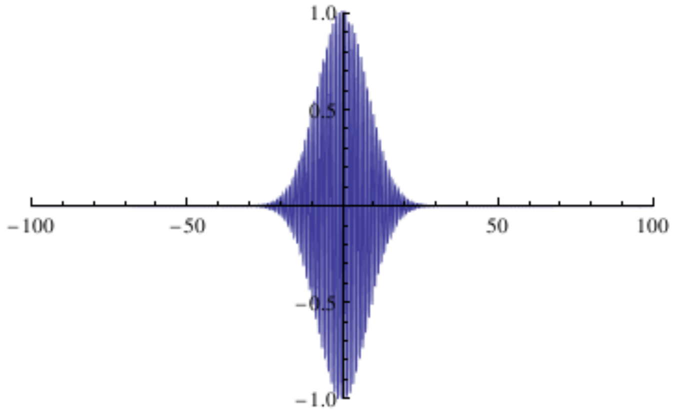
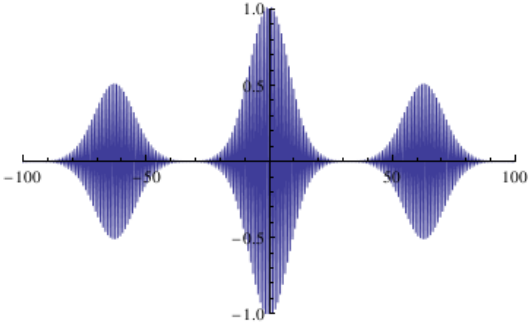


Result with ε = 0.10; approximating the integral by the sum of 480 k values:



Result with ε = 0.20; approximating the integral by the sum of 240 k values: