# <u>Maxwell's Equations in Matter</u>

Start with the fundamental equations

(1) $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$

(2) $\nabla\cdot\mathbf{B} = 0$

(3) $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$

(4) $\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\, \partial\mathbf{E}/\partial t$

where $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$ satisfy the continuity equation,

$$\nabla\cdot\mathbf{J} = -\partial\rho/\partial t$$

(conservation of charge)

Now consider the effects of macroscopic matter; i.e., matter with many atoms ($6 \times 10^{23}$ per mole).

We'll treat the charge and current in atoms by averaging --- a good approximation because N is so large;

$$\text{error} \sim 1/\sqrt{N} \sim 1/\sqrt{(6\times10^{23})} \sim 1/10^{12}$$

Two of Maxwell's equations do not depend on matter:

(2) $\nabla\cdot\mathbf{B} = 0$

*(primary magnetic monopoles do not exist)*

(3) $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$

*(EM induction is a field effect)*

The other two depend on sources, i.e., matter, for which we'll make macroscopic approximations.

## Charge density

$$\rho_{micro}(\mathbf{x},t)=\sum_{i=1..N} e_i \ \delta^3(\mathbf{x}-\mathbf{x}_i(t))$$

or, in quantum theory,

$$\rho_{micro}(\mathbf{x},t)=\sum_{i=1..N} e_i \ |\psi_i(\mathbf{x},t)|^2$$

where $N \sim 6 \times 10^{23}$

Now define the *macroscopic average*,

$$\rho(\mathbf{x},t) = (1/\delta V) \int_{\delta V} \rho_{micro}(\mathbf{x}',t)d^3x'$$

where $\delta V$ is a volume around the point **x**, both ***large*** and ***small*** :

-- ***large*** compared to an atom

-- ***small*** compared to the full system

Recall the methods for dielectric materials.

$$\rho(\mathbf{x},t) = \rho_{free}(\mathbf{x},t) + \rho_{bound}(\mathbf{x},t)$$

free charge is charge that is not bound in atoms or molecules;
bound charge is charge that is confined to an atom or molecule.

Review:

$$\rho_{bound}(\mathbf{x},t) = -\nabla\cdot\mathbf{P}$$

**P** = polarization = electric dipole moment density

$$\mathbf{P}(\mathbf{x},t) = n(\mathbf{x},t) \ \langle\mathbf{p}\rangle$$

## Gauss's Law for the electric field $\mathbf{E}(x,t)$

$$\nabla\cdot\mathbf{E} = \rho_{free}/\varepsilon_0 + \rho_{bound}/\varepsilon_0 = \rho_{free}/\varepsilon_0 - \nabla\cdot\mathbf{P}/\varepsilon_0$$

$$\nabla\cdot(\varepsilon_0\mathbf{E} + \mathbf{P}) = \rho_{free}$$

Define the displacement field $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ ; then
$\nabla\cdot\mathbf{D} = \rho_{free}$.

But we will need a "constitutive equation" to relate **E(x, t)** and **D(x,t)**.

For linear dielectrics,

$\mathbf{D} = \varepsilon\mathbf{E}$ or $\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}$

(*permittivity* or *susceptibility*)

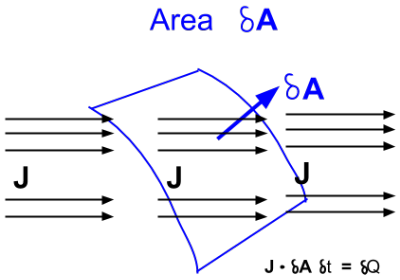
All of this is familiar for static fields. It is a good enough approximation for time dependent fields, unless photon energy > ~keV.
However, even for low photon energies, $\varepsilon$ may depend on frequency.

## Electric current density

The fundamental field equation is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \, \partial \mathbf{E}/\partial t$$

Now, what is current density $\mathbf{J}(\mathbf{x},t)$?




*Any process that moves charge contributes to the current density.*

Again, we separate free current and bound current.

$$\mathbf{J} = \mathbf{J}_{free} + \mathbf{J}_{magnetization} + \mathbf{J}_{polarization}$$

In Chapter 9 we considered bound current that is related to the magnetization $\mathbf{M}(\mathbf{x},t)$,

$$\mathbf{J}_{magn.} = \nabla \times \mathbf{M} \; ; \quad \mathbf{M}(\mathbf{x},t) = n(\mathbf{x},t) \langle \boldsymbol{m} \rangle$$

This is current inside an atom (or molecule) which gives the atom a magnetic dipole moment ***m***.

But now, for time-dependent systems, there is another kind of bound current related to the polarization $\mathbf{P}(\mathbf{x},t)$,

$$\mathbf{J}_{pol.} = \partial \mathbf{P}/\partial t; \quad \mathbf{P}(\mathbf{x},t) = n(\mathbf{x},t) \langle \mathbf{p} \rangle$$

**Result:** $\mathbf{J} = \mathbf{J}_{free} + \nabla \times \mathbf{M} + \partial \mathbf{P}/\partial t$

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**Theorem.** $\nabla \cdot \mathbf{J} = -\partial \rho/\partial t$

(continuity equation)

**Proof.**

$$\nabla \cdot \mathbf{J} = \nabla \cdot \mathbf{J}_{free} + \nabla \cdot (\nabla \times \mathbf{M}) + \nabla \cdot (\partial \mathbf{P}/\partial t)$$

$$= -\,\partial \rho_{free}/\partial t + 0 + \partial(-\rho_{bound})/\partial t$$

$$= -\,\partial \rho/\partial t$$

Q.E.D.

*__The Ampere-Maxwell equation in the presence of matter.__*

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0\, \partial \mathbf{E}/\partial t$$

$$= \mu_0(\mathbf{J}_{free} + \nabla \times \mathbf{M} + \partial \mathbf{P}/\partial t) + \mu_0 \varepsilon_0\, \partial \mathbf{E}/\partial t$$

$$\nabla \times (\mathbf{B}/\mu_0 - \mathbf{M}) = \mathbf{J}_{free} + \partial(\varepsilon_0 \mathbf{E} + \mathbf{P})\,/\partial t$$

$$(4) \quad \nabla \times \mathbf{H} = \mathbf{J}_{free} + \partial \mathbf{D}\,/\partial t$$

where $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ and $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$

displacement current $= \partial \mathbf{D}\,/\partial t$

Constitutive equations for linear magnetic materials (diamagnetic or paramagnetic):

$$\mathbf{B} = \mu \mathbf{H} \quad \text{or} \quad \mathbf{M} = \mu_0 \chi_m \mathbf{B}$$

**Maxwell's Equations**, including the atomic effects of magnetization and polarization in macroscopic matter...

(1) $\nabla \cdot \mathbf{D} = \rho_{free}$

(2) $\nabla \cdot \mathbf{B} = 0$

(3) $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$

(4) $\nabla \times \mathbf{H} = \mathbf{J}_{free} + \partial \mathbf{D} / \partial t$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon\ \mathbf{E}$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu\ \mathbf{H}$$

*Consequences for energy density and energy flux (see Section 11.X)*

The energy density, which includes both field energy and material energies, is

$$u = ½\ (\ \mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\ )$$

The energy flux is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

**Exercise 11.X.** The continuity equation for energy conservation is

$$\nabla \cdot \mathbf{S} = -\partial \mathbf{B}/\partial t - \mathbf{J}_{free}\cdot\mathbf{E}$$