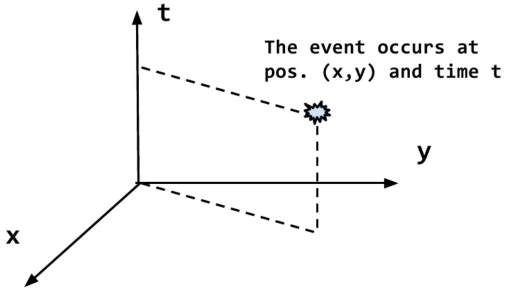
Electromagnetism and Relativity

(Chapter 12)

We live in a 4-dimensional spacetime. Events (e.g., measurements) are observed with respect to coordinates; space (x,y,z) and time (t).



This frame of reference is called an inertial frame if the "law of inertia" holds true in terms of these coordinates.

4-vector notation

$$x^{\mu} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

which can be confusing: sometimes x^{μ} means a 4-vector; but other times x^{μ} means the μ component of a vector.

Recall 3-vectors:

$\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

x^i is a notation for the i - th component of $\mathbf{x}$

$x^1 = x\,;\ \ x^2 = y\,;\ \ x^3 = z$

$\mu, \upsilon, \lambda, \ldots$ greek letters = 0, 1, 2, 3

i, j, k, ... roman letters = 1,2,3

Einstein's Postulates of Special Relativity

(1) The laws of physics have the same form in all inertial frames.

(2) The speed of light in free space is the same in all inertial frames; $c = 2.998 \times 10^8$ m/s.

Lorentz transformations

(The equations were first published by Lorentz; they were given a more profound interpretation by A. Einstein.)

Consider two inertial frames,

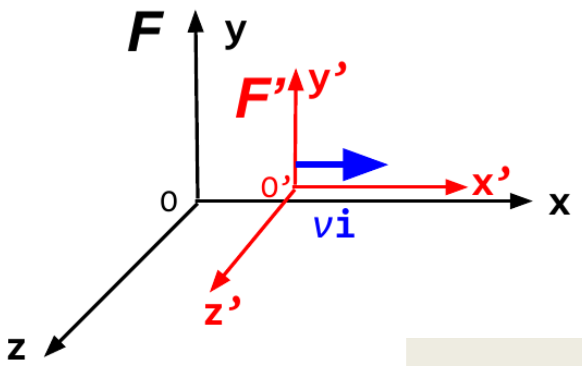
$\mathfrak{F}$: (ct, x, y, z)

$\mathfrak{F}'$: (ct', x', y', z')

Assume that $\mathfrak{F}'$ moves with velocity $v\mathbf{i}$ with respect to $\mathfrak{F}$.

(Later we'll generalize the results for $\mathbf{v}$ in an arbitrary direction.)

$\mathfrak{F}$ and $\mathfrak{F}'$



The position of O' (the origin of F') is $x_{O'}$ with respect to F; $x_{O'}(t) = v\,t$.

Exercise.

Derive the transformation $x^\mu \rightarrow x'^\mu$.

I.e., given the coordinates of an event in $\mathfrak{F}$, what are the coordinates of the same event in $\mathfrak{F}'$?

Assume it is linear, so

$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu\, x^\nu$ (matrix-vector notation)

$$ct' = \Lambda^0{}_0\, ct + \Lambda^0{}_1\, x + \Lambda^0{}_2\, y + \Lambda^0{}_3\, z$$
$$x' = \Lambda^1{}_0\, ct + \Lambda^1{}_1\, x + \Lambda^1{}_2\, y + \Lambda^1{}_3\, z$$
$$y' = \Lambda^2{}_0\, ct + \Lambda^2{}_1\, x + \Lambda^2{}_2\, y + \Lambda^2{}_3\, z$$
$$z' = \Lambda^3{}_0\, ct + \Lambda^3{}_1\, x + \Lambda^3{}_2\, y + \Lambda^3{}_3\, z$$

(The matrix notation is so much simpler. We will need to get used to ***4-vector notations***; this is also called ***tensor analysis.***)

Galilean Relativity

$$ct' = ct \qquad (1)$$
$$x' = x - vt \qquad (2)$$
$$y' = y$$
$$z' = z$$

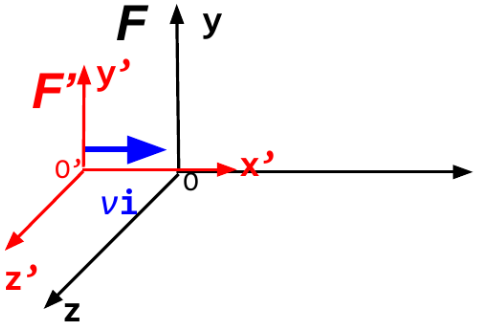
(1) there is a universal time .
(2) x'=0 means x = vt, as specified .
Spatial axes just translate without any scale change.
So what's wrong with this? The speed of light would not be constant.

Special Relativity
Try this …

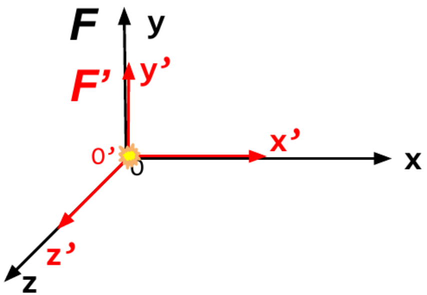
$$ct' = A_1\, ct + A_2\, x$$
$$x' = A_3\, (x - vt)$$
$$y' = y$$
$$z' = z$$

Note: **x'** = 0 means **x** = **vt**, as specified.

… and determine A_1, A_2, A_3 to satisfy the postulate of the ***absolute speed of light.***

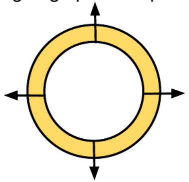


Suppose a flash of light occurs at the origin O' at time t'=0.



The light propagates outward at speed c, ***in either frame of reference.***

outgoing spherical pulse



- W.R.T. frame ***F'***
  $(x'^2 + y'^2 + z'^2)^{½} = ct'$
  or
  $x'^2 + y'^2 + z'^2 = c^2t'^2$ (1)
- W.R.T. frame ***F***
  $x^2 + y^2 + z^2 = c^2t^2$ ***(same c!)*** (2)

Substitute the Lorentz transformation into Eq. (1) and compare the result to Eq. (2).

$$A_3^2(x-vt)^2 + y^2 + z^2 = (A_1\, ct + A_2\, x)^2$$

so …

$$c^2t^2 - x^2 = (A_1\, ct + A_2\, x)^2 - A_3^2\,(x - vt)^2$$

$$= (A_1^2 - A_3^2\, v^2/c^2)\, c^2t^2 + (A_1 A_2 + A_3^2\, v/c)\, 2\, ct\, x + (A_2^{2.} - A_3^2)\, x^2$$

Thus, we require,

$$A_1^2 - A_3^2\, v^2/c^2 = 1$$
$$A_1\ A_2 + A_3^2\, v/c = 0$$
$$A_2^2 - A_3^2 = -1$$

Exercise: verify that the solution of these equations is

$$A_1 = A_3 = (1-v^2/c^2)^{-1/2}$$

and $A_2 = -(v/c)(1-v^2/c^2)^{-1/2}$

The Lorentz transformation $\mathcal{F} \to \mathcal{F}'$

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\boxed{\mathcal{F}' \leftarrow \mathcal{F}}$$

The reverse transformation $\mathcal{F}' \to \mathcal{F}$

$$ct = \gamma\left(ct' + \frac{v}{c}x'\right)$$
$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$

$$\boxed{\mathcal{F} \leftarrow \mathcal{F}'}$$

Why? Because $\mathcal{F}$ moves with velocity $-v\hat{i}$ w.r.t. $\mathcal{F}'$.

($\vec{x} = 0$ means $\vec{x}' = -v\hat{i}\,t'$.) Origin of $\mathcal{F}$.

Change the sign of v.

For an arbitrary direction of the relative velocity $\vec{v}$

Say $\mathcal{F}'$ moves with velocity $\vec{v}$ w.r.t. the frame $\mathcal{F}$. Then

$$ct' = \gamma\left(ct - \frac{v}{c}x_{\parallel}\right)$$
$$x'_{\parallel} = \gamma(x_{\parallel} - vt)$$
$$\vec{x}'_{\perp} = \vec{x}_{\perp}$$

$$\gamma = 1/\sqrt{1-v^2/c^2}$$

The Lorentz contraction effect. A meter stick moves with velocity v along the x axis of inertial frame ***F.***
Calculate the distance between the ends of the stick at a given time t (*i.e., for the same time t at both ends*).
In the rest frame (***F'***) the length is D' = 1 meter.

$D' = \gamma\,(\Delta x - v\,\Delta t)$

Thus $\Delta x = D'/\gamma = (1 - v^2/c^2)^{1/2}\,D'$

If v = 0.1 c then Δx = 0.995 m; if v = 0.9 c then Δx = 0.436 m.