Particle Dynamics
in Special Relativity

In classical dynamics (Newton's laws of motion) there is a universal time coordinate t; then

x(t) = particle position

u(t) = d**x**/dt = velocity

p = m **u** (momentum)

and d**p**/dt = **F** (Newton's second law)

The equations are written in terms of ***3-vectors***; the time is universal.

In relativistic dynamics, we must follow the particle in the 4-dimensional spacetime,

$x^\mu(\tau)$ = 4-position, a function of ***proper time*** τ.

Proper time (from the French "propre") is the time coordinate in a frame of reference (***F''***) co-moving with the particle. $d\tau = dt''$

Then we have

$\eta^\mu = dx^\mu / d\tau$ = 4-velocity

$p^\mu = m\,\eta^\mu$ = 4-momentum

$dp^\mu / d\tau = K^\mu$ = the "Minkowski force", a 4-vector

We must express the dynamical equations in terms of 4-vectors, because that's the only way to ensure that the laws of physics (equations) are the same in all frames of reference.

Proper time (from the French "propre")

- τ is a scalar
- In an arbitrary frame of reference

(***F***)

$$c\,(d\tau) = \sqrt{c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}$$

or,

$$c^2\,(d\tau)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

or,

$$c^2\,(d\tau)^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$$

(* the Einstein summation convention)

- $d\tau$ is a scalar! $d\tau' = d\tau$

I.e., $d\tau$ has the same value in all inertial frames. Transformation: $d\tau' = d\tau$

The metric tensor

$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

$g_{00} = 1$; $g_{11} = g_{22} = g_{33} = -1$

$g_{\mu\nu} = 0$ if μ not equal to ν

Theorem : $d\tau$ is a scalar.

Proof: Consider an arbitrary inertial frame ***F.***

We define proper time,

$c^2 (d\tau)^2 = c^2 (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$

Now consider the ***co-moving*** frame ***F''.***

The displacement 4-vector in this frame is

dt'', with $dx''=0$, $dy''=0$, $dz''=0$;

Apply the Lorentz transformation from ***F''*** to ***F***:

(W.L.O.G., assume the relative velocity is v **i**)

$c(dt) = \gamma \{ c(dt'') + (v/c)\, dx'' \} = \gamma\, c(dt'')$;

$dx = \gamma \{ dx'' + v\, dt'' \} = \gamma\, v\, dt''$;

$dy = dy'' = 0$

$dz = dz'' = 0$

Thus

$c^2 (d\tau)^2 = \gamma^2 \{ c^2 - v^2 \} (dt'')^2$

But the right-hand side is just $c^2(dt'')^2$.

So $d\tau = dt''$ for any inertial frame ***F.***

Q.E.D.

Scalars and Vectors

- $d\tau$ is a scalar (i.e., invariant)
- x^μ is a vector (by definition)
- anything that transforms in the same way as x^μ is a vector (by definition)
- therefore dx^μ is a vector; it's just $x_2^\mu - x_1^\mu$
- $\eta^\mu = dx^\mu/d\tau$ is a vector
- $p^\mu = m\, \eta^\mu$ is a vector (m is the rest mass, invariant)
- $dp^\mu/d\tau$ is a vector
- So if we have a dynamical equation $dp^\mu/d\tau = K^\mu$, then K^μ must be a vector (the Minkowski force)

Energy and momentum of a particle

Exercise. Express these 4-vectors in terms of the particle 3-velocity $\mathbf{u}$: η^μ and p^μ

Define particle velocity $\mathbf{u}$ as it would be measured by an observer in frame $\boldsymbol{F}$:

$\mathbf{u} = d\mathbf{x}/dt$ (a 3-vector)

Now recall the 4-vector position: X^μ

$$\begin{vmatrix} ct \\ x \\ y \\ z \end{vmatrix}$$

<u>$d\tau$</u>

$$c^2\,(d\tau)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

$$c^2\,(d\tau)^2 = c^2(dt)^2 - u_x{}^2(dt)^2 - u_y{}^2(dt)^2 - u_z{}^2(dt)^2$$

$$c^2\,(d\tau)^2 = (c^2 - u^2)(dt)^2$$

$d\tau = dt\ (1-u^2/c^2)^{1/2}$ (the time dilation formula)

<u>$\eta^\mu = dx^\mu/d\tau$</u>

$$\eta^\mu = \{\eta^0, \boldsymbol{\eta}\} = \{c\ dt/d\tau,\ d\mathbf{x}/d\tau\}$$

$$= \{c\ /(1-u^2/c^2)^{½},\ \mathbf{u}\ /(1-u^2/c^2)^{1/2}\}$$

$$\eta^\mu = \quad 1/(1-u^2/c^2)^{½} \begin{vmatrix} c \\ u_x \\ u_y \\ u_z \end{vmatrix}$$

<u>$p^\mu = m\ \eta^\mu$</u>

$$p^\mu = \{p^0,\ \mathbf{p}\}$$

$p^0 = mc\ /(1-u^2/c^2)^{½}$ = particle energy /c

$\mathbf{p} = mu\ /(1-u^2/c^2)^{½}$ = particle momentum

4-momentum

In classical physics, momentum $\vec{p} = m\vec{u} = m\dfrac{d\vec{x}}{dt}$

In relativity we use 4-momentum

$$p^\mu = m\frac{dx^\mu}{d\tau} = \begin{pmatrix} p^0 \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} \quad \text{and} \quad p^0 = \frac{mc}{\sqrt{1-u^2/c^2}}$$

↑ particle momentum

↑ particle energy/c

$$E = cp^0 = \frac{mc^2}{\sqrt{1-u^2/c^2}}$$ particle energy

For velocity $\vec{u} = 0$, $E = mc^2$.

Dynamics — the Minkowski force

$$\frac{dp^\mu}{d\tau} = K^\mu \quad \text{(which comes from some interaction theory)}$$

- **Space components**

An observer defines the force on a particle by $d\vec{p}/dt = \vec{F}$.

(def. of $\vec{F}$)

Now $\vec{K} = \dfrac{d\vec{p}}{d\tau} = \dfrac{d\vec{p}}{dt}\dfrac{dt}{d\tau} = \dfrac{\vec{F}}{\sqrt{1-u^2/c^2}}$

If $\vec{F} = 0$ then $\vec{p}$ = constant.

- **The time component**

$$K^0 = \frac{dp^0}{d\tau} = \frac{dp^0}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}}\frac{dp^0}{dt}$$

and $p^0 = \dfrac{mc}{\sqrt{1-u^2/c^2}}$.

$$K^0 = \frac{mc}{\sqrt{1-u^2/c^2}}\left(\frac{-1}{2}\right)\frac{-2\vec{u}\cdot d\vec{u}/dt}{(1-u^2/c^2)^{3/2}c^2} = \frac{mc\,\vec{u}\cdot d\vec{u}/dt}{(1-u^2/c^2)^2\,c^2}$$

Compute:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}}\right) = m\left\{\frac{d\vec{u}/dt}{\sqrt{1-u^2/c^2}} + \frac{\vec{u}\,(\vec{u}\cdot d\vec{u}/dt)}{(1-u^2/c^2)^{3/2}c^2}\right\}$$

Thus $\vec{u}\cdot\vec{F} = m\dfrac{\vec{u}\cdot(d\vec{u}/dt)}{(1-u^2/c^2)^{3/2}}$

Result $K^0 = \dfrac{\vec{u}\cdot\vec{F}}{c\sqrt{1-u^2/c^2}}$

$\boxed{\text{Energy}}$ $\dfrac{dp^0}{dt} = K^0\sqrt{1-u^2/c^2} = \dfrac{\vec{u}\cdot\vec{F}}{c} = \dfrac{\text{work/time}}{c}$

To be consistent with the conservation of energy, p^0 must be E/c ; $E = cp^0 = \dfrac{mc^2}{\sqrt{1-u^2/c^2}}$.