<u>The Electromagnetic Field Tensor</u> - part I

$F^{\mu\nu}(x)$

Consider an arbitrary inertial frame

$\boldsymbol{F}$ **:** { ct, x, y, z }

How do we define the fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$?

The force on a test charge q is

$$d\mathbf{p}/dt = \mathbf{F} = q\,\mathbf{E} + q\,\mathbf{u} \times \mathbf{B}$$

where

$$\mathbf{u} = \text{velocity} = d\mathbf{x}/dt.$$

Recall: $\mathbf{p} = m\mathbf{u} \,/\, \sqrt{1-u^2/c^2}$

Now, by the principles of special relativity (*the laws of physics are the same in all inertial frames*) we should write the equations in <u>covariant form</u>; i.e., write the equations only in terms of Lorentz scalars, vectors and tensors.

Mathematics of Tensor Analysis

Definitions and Notations

Contravariant vectors : x^μ , A^μ , ...

Covariant vectors : x_μ , A_μ , ...

These are related by

$$x_\mu = g_{\mu\nu} x^\nu \quad ; \quad \text{or,} \quad A_\mu = g_{\mu\nu} A^\nu$$

where we use the Einstein summation convention (so the sum over ν from 0 to 3 is implied).

Here $g_{\mu\nu}$ is the metric tensor

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1);$$

note that

$$A_0 = A^0$$

$$A_1 = -A^1 , \; A_2 = -A^2 , \; A_3 = -A^3$$

"Raising and lowering the index"

$(a_0 , a_1 , a_2 , a_3) = (a^0 , -a^1 , -a^2 , -a^3)$

Or, equivalently, $a_\mu = g_{\mu\nu} a^\nu$ or $a^\mu = g^{\mu\nu} a_\nu$

where $g^{\mu\nu} = g_{\mu\nu}$.

Theorem 1. If A^μ and B^μ are Lorentz vectors (contravariant) then $A^\mu B_\mu$ is a scalar.

(* we always use the Einstein summation convention, so $A^\mu B_\mu$ means that we sum over μ from 0 to 3. *)

Note this tricky important point:

$A^\mu B_\mu$ has no index.

μ is summed from 0 to 3.

$A^\mu B_\mu$ means the sum of four terms.

Proof #1.

Consider 2 inertial frames, ***F*** and ***F'*** & relative velocity **v** .

Relative to frame F we have

$$A^\mu B_\mu = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3$$

(do you see why?)

$$= A^0 B^0 - A_\parallel B_\parallel - \mathbf{A}_\perp \cdot \mathbf{B}_\perp$$

Now consider the Lorentz transformation...

$$A'^\mu B'_\mu = A'^0 B'^0 - A'_\parallel B'_\parallel - \mathbf{A'}_\perp \cdot \mathbf{B'}_\perp$$

$$= \gamma[A^0 - (v/c)A_\parallel]\,\gamma[B^0 - (v/c)B_\parallel]$$

$$-\gamma[A_\parallel - (v/c)A^0]\,\gamma[B_\parallel - (v/c)B^0] - \mathbf{A}_\perp \cdot \mathbf{B}_\perp$$

$$= \gamma^2 A^0 B^0 (1 - v^2/c^2) - \gamma^2 A_\parallel B_\parallel (1 - v^2/c^2) - \mathbf{A}_\perp \cdot \mathbf{B}_\perp$$

$$= A^0 B^0 - A_\parallel B_\parallel - \mathbf{A}_\perp \cdot \mathbf{B}_\perp$$

$$= A^\mu B_\mu$$

Q. E. D.

Proof #2.

$$A'^\mu B'_\mu = g_{\mu\nu} A'^\mu B'^\nu$$

($\Lambda^\mu{}_\rho$ = the Lorentz transformation matrix)

$$= g_{\mu\nu} \quad \Lambda^\mu{}_\rho A^\rho \quad \Lambda^\nu{}_\sigma B^\sigma$$

(Einstein summation convention for μ, ν, ρ, σ)

$$= g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \; A^\rho B^\sigma$$

Exercise: Prove that $g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}$.

The metric tensor is the same in all inertial frames; diag(1, -1, -1, -1).

So... $A'^\mu B'_\mu = g_{\rho\sigma} A^\rho B^\sigma = A^\rho B_\rho$ (Q. E. D.)

Do you get this?

$A^\mu B_\mu$ does not depend on μ because μ is summed from 0 to 3 by the Einstein summation convention.

$$A^\mu B_\mu = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3$$

$$= A^\nu B_\nu = A^\rho B_\rho = A^\xi B_\xi$$

$\Lambda \; g \; \Lambda = g$

Theorem 2. If A^μ is a Lorentz vector and $C^{\mu\nu}$ is a Lorentz tensor, then $C^{\mu\nu}A_\nu$ is a Lorentz vector.

Proof. What do we need to prove? We need to prove that $C^{\mu\nu}A_\nu$ transforms in the same was as x^μ ; i.e., (remember, $x'^\mu = \Lambda^\mu{}_\rho \, x^\rho$)

$$C'^{\mu\nu}A'_\nu = \Lambda^\mu{}_\rho \, C^{\rho\nu}A_\nu$$

(N. B. : the Einstein summation convention applies to ρ and ν)

$$C'^{\mu\nu}A'_\nu = g_{\nu\lambda} \, C'^{\mu\nu}A'^\lambda = g_{\nu\lambda} \, \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma C^{\rho\sigma} \, \Lambda^\lambda{}_\kappa A^\kappa$$

$$= \Lambda^\mu{}_\rho \, g_{\sigma\kappa} \, C^{\rho\sigma} A^\kappa = \Lambda^\mu{}_\rho \, C^{\rho\sigma} A_\sigma$$ **(Q. E. D.)**

Do you get this?
Ignoring the indices, this is how it goes…
$C'A' = g\,C'\,A' = g\,\Lambda\Lambda C\,\Lambda A$
$= \Lambda\,\Lambda g \Lambda\,CA = \Lambda\,g\,CA = \Lambda\,CA$;
… but make sure the indices work out correctly!

Summary and generalizations

Contraction of …

contravariant vector and covariant vector → scalar

contravariant tensor and covariant vector → contravariant vector

tensor of rank n and tensor of rank m → tensor of rank |n-m|

rank 1 $\square$ rank 1 = rank 0 (i.e., V $\square$ V = S)

rank 2 $\square$ rank 1 = rank 1 (i.e., T $\square$ V = V)

In general, $T_1 \square T_2 = T_3$ where the rank of T_3 is $n_3 = n_1 + n_2$ − # of contracted indices

There are also tensors with mixed contravariant and covariant indices: $T_{\alpha\beta\gamma}{}^{\lambda\mu\nu}$

That is ***the algebra of tensors***.

The calculus of tensors

Theorem 4. The differential operator $\partial/\partial x^\mu$, which we sometimes denote by ∂_μ, transforms as a covariant vector.

Proof. Let $\varphi(x)$ be a scalar function of x^μ.

Now consider $\Phi_\mu \equiv \partial_\mu \varphi$.

According to the theorem, it is a covariant vector. Or, equivalently, Φ^μ is a contravariant vector. That's what we have to prove.

Example: Let $\varphi = g_{\rho\sigma} x^\rho x^\sigma$ (a scalar);

then $\partial_\mu \varphi = g_{\mu\sigma} x^\sigma + g_{\rho\mu} x^\rho$

$= 2\, g_{\mu\lambda}\, x^\lambda = 2\, x_\mu$

... a covariant vector, as claimed.

Now watch carefully ...

$$\Phi'^\mu = g^{\mu\nu}\, \Phi'_\nu = g^{\mu\nu}\ \partial/\partial x'^\nu\, \varphi(x')$$

$$= g^{\mu\nu}\, \partial/\partial x'^\nu\, \varphi(x) \qquad \{\text{ because } \varphi \text{ is a scalar }\}$$

$$= g^{\mu\nu}\, \partial\varphi/\partial x^\rho \quad \partial x^\rho / \partial x'^\nu$$

$$\{\text{ sum over } \rho \text{ is implied! }\}$$

$$= g^{\mu\nu}\, \Phi_\rho \quad [\Lambda^{-1}]^\rho{}_\nu$$

$$\{x' = \Lambda x \text{ implies } x = \Lambda^{-1} x'\}$$

$$= \Lambda^\mu{}_\sigma\, g^{\sigma\rho} \quad \Phi_\rho$$

$$(\text{recall: } \Lambda g \Lambda = g)$$

$$= \Lambda^\mu{}_\sigma\, \Phi^\sigma$$

(Q. E. D.)

Consequences and generalizations

$\partial_\mu \varphi = \Phi_\mu$ vector

$\partial_\mu G_\nu = T_{\mu\nu}$ tensor

Differentiation produces tensors from vectors.