Correction

We'll use this convention for the metric tensor:

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \quad .$$

In other words, $A_\mu = g_{\mu\nu} A^\nu$ with

$A_0 = - A^0$; $A_i = + A^i$ (i = 1 2 3)

(equation 12.24)

For example,

$$c^2 (d\tau)^2 = c^2 (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

$$= - dx^\mu dx_\mu$$

All the results derived in the last lecture are still true with this notation, because the change is only a change of sign from (+−−−) to (−+++).

The electromagnetic field tensor -- part 2

$F^{\mu\nu}$

Einstein's postulate of relativity: the laws of physics are the same in all inertial frames.

Therefore we can, and should, write the laws of physics (i.e., equations) in covariant form.

Theorem. An equation is covariant if it is written in terms of Lorentz tensors, vectors and scalars.

Review tensor analysis

Scalar: invariant w.r.t. Lorentz transformations;
example $-dx^{\mu}\, dx_{\mu} = c^2(d\tau)^2$

Vector: $V'^{\mu} = \Lambda^{\mu}{}_{\rho}\, V^{\rho}$;
example, x^{μ} .

Tensor: $T'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\, \Lambda^{\nu}{}_{\xi}\, T^{\rho\xi}$;
example, $g^{\mu\rho}\, \partial_{\rho}\, A^{\nu}$.

Proof:

Consider a theoretical equation, w. r. t. inertial frame ***F***,

$$T_1^{\mu\nu} = T_2^{\mu\nu}$$

What is the corresponding equations w. r. t. another inertial frame ***F'***?

$$T'^{\mu\nu}_1 = \Lambda^\mu_\rho \, \Lambda^\nu_\sigma \, T_1^{\rho\sigma}$$

$$= \Lambda^\mu_\rho \, \Lambda^\nu_\sigma \, T_2^{\rho\sigma}$$

$$= T'^{\mu\nu}_2$$

Q. E. D. The equation is covariant… it takes the same form in all inertial frames.

Particle dynamics in covariant form

Define 4-momentum $p^\mu = m\, dx^\mu/d\tau$

The equations of motion is

$$dp^\mu /d\tau = K^\mu \ .$$

Some other theory must determine the Minkowski force, K^μ .

What is K^μ for a charged particle in electric and magnetic fields?

We know,

$$d\mathbf{p} /dt = q\, [\, \mathbf{E} + \mathbf{u} \times \mathbf{B}\,]$$

$$\mathbf{u} = d\mathbf{x} /dt$$

$$\text{and} \ \ \mathbf{p} = m\mathbf{u} /\sqrt{1 - u^2/c^2}$$

Calculation of $dp^\mu / d\tau = K^\mu$.

Calculation of $dp^\mu/d\tau$

$$\frac{dp^\mu}{d\tau} = \frac{dp^\mu}{dt}\frac{dt}{d\tau}$$

where $d\tau = \sqrt{(dt)^2 - d\vec{x}\cdot d\vec{x}/c^2}$

$\hookrightarrow d\vec{x} = \vec{u}\,dt$

$$d\tau = dt\sqrt{1-u^2/c^2}$$

$$\frac{dp^\mu}{d\tau} = \frac{dp^\mu}{dt}\frac{1}{\sqrt{1-u^2/c^2}} = \frac{dp^\mu}{dt}\gamma$$

Calculation of $dp^\mu / d\tau = K^\mu$.

Now consider $\underline{\underline{\mu = i}}$ $(i = 1, 2, 3)$

$$\frac{dp^i}{d\tau} = \frac{dp^i}{dt}\gamma$$

$$K^i = q[E^i + (\vec{u}\times\vec{B})^i]\gamma$$

$$= q[E^i + \epsilon_{ijk} u^j B^k]\gamma$$

Now the case $\underline{\underline{\mu = 0}}$ $\quad p^0 = E/c$

$$K^0 = \frac{dp^0}{d\tau} = \frac{dE/c}{dt}\gamma = \frac{\gamma}{c}\vec{F}\cdot\vec{u}$$

$$= \frac{\gamma}{c} q\vec{E}\cdot\vec{u} = \frac{\gamma}{c} q E^j u^j$$

Conservation of energy; work

Now take a look at this vector

$$L^{\mu} \equiv q\, F^{\mu\nu}\, \eta_{\nu} \quad .$$

($F^{\mu\nu}$ must be a tensor!)

$$K^{i} = q\gamma \{ E^{i} + \epsilon_{ijk}\, u^{j} B^{k} \}$$

$$K^{0} = q\gamma\, \frac{E^{j}}{c}\, u^{j}$$

$$L^{\mu} = qF^{\mu\nu}\eta_{\nu} = q\,(-F^{\mu 0}\eta^{0} + F^{\mu j}\eta^{j})$$

$$L^{0} = q\{ -F^{00} c\gamma + F^{0j} u^{j}\gamma \}$$

$$= \gamma q \{ -F^{00} c + F^{0j} u^{j} \}$$

$\longrightarrow$ Make $L^{0} = K^{0}$; requires $F^{00} = 0$ and $F^{0j} = E^{j}/c$

Aside calculation:

4 - velocity

$n^{0} = c\,/(1-u^{2}/c^{2})^{1/2}$

$n^{i} = u^{i}\,/(1-u^{2}/c^{2})^{1/2}$

$$L^{i} = q\{ -F^{i0}\eta^{0} + F^{ij}\eta^{j} \}$$

$$= q\{ -F^{i0} c\gamma + F^{ij} u^{j}\gamma \}$$

$$= q\gamma \{ -F^{i0} c + F^{ij} u^{j} \}$$

$\longrightarrow$ Make $L^{i} = K^{i}$; requires $F^{i0} = -E^{i}/c$ and $F^{ij} = \epsilon_{ijk} B^{k}$

<u>Result</u> $K^{\mu} = q\, F^{\mu\nu}\eta_{\nu}$ and $F^{\mu\nu}$ must be a tensor

The electromagnetic field tensor

(1) $F^{\mu\nu}$ must be a tensor.

(2) $F^{00} = 0$; $F^{0i} = - F^{i0} = E^i / c$; $F^{ij} = \varepsilon_{ijk} B^k$.

(3)

	v	0	1	2	3
μ					
0		**0**	E_x / c	E_y / c	E_z / c
1		$-E_x / c$	**0**	B_z	$-B_y$
2		$-E_y / c$	$-B_z$	**0**	B_x
3		$-E_z / c$	B_y	$-B_x$	**0**