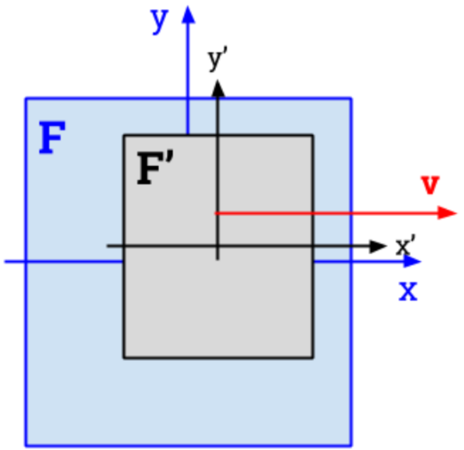
# ***Electric and magnetic field transformations***

Consider inertial frames *F'* and *F*. Suppose we know the electric and magnetic fields, **E'**(**x'**,t') and **B'**(**x'**,t'), as observed in frame *F'*.

Then what are the electric and magnetic fields, **E**(**x**,t) and **B**(**x**,t), as observed in frame *F*?

Suppose *F'* moves with velocity **v** relative to *F*. WLOG, take **v** to be in the x direction; **v** = v **i**.

Picture:




Fields are present, which are measured, by the forces that they produce, with respect to the two different frames of reference, ***F*** and ***F'***.

## ***Electric and magnetic field transformations***

To figure out **E**(**x**,t) and **B**(**x**,t):

(1) We already know the field tensor $F'^{\mu\nu}$ for the inertial frame *F'*.

(2) We know how to transform the field tensor,

$$F^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\,\Lambda^{\nu}{}_{\sigma}\,F'^{\rho\sigma}$$

where $\Lambda^{\mu}{}_{\rho}$ is the Lorentz transformation matrix.

(3) Finally deconstruct $F^{\mu\nu}$ into **E**(**x**,t) and **B**(**x**,t).

(1)

$$F'^{\mu\nu} = \begin{matrix} 0 & E'_x/c & E'_y/c & E'_z/c \\ -E'_x/c & 0 & B'_z & -B'_y \\ -E'_y/c & -B'_z & 0 & B'_x \\ -E'_z/c & B'_y & -B'_x & 0 \end{matrix}$$

(2) $x^{\mu} = \Lambda^{\mu}{}_{\rho}\, x'^{\rho}$ ; $F^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\,\Lambda^{\nu}{}_{\sigma}\,F'^{\rho\sigma}$

$$\Lambda^{\mu}{}_{\rho} = \begin{matrix} \gamma & \gamma v/c & 0 & 0 \\ \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

(3)

$$\{E_x, E_y, E_z\} = \{c\,F^{01}, c\,F^{02}, c\,F^{03}\}$$
$$\{B_x, B_y, B_z\} = \{F^{23}, F^{31}, F^{12}\}$$

**$\mu\nu = 01$**

$F^{01} = \Lambda^0{}_\rho \Lambda^1{}_\sigma F'^{\rho\sigma} = \Lambda^0{}_0 \Lambda^1{}_1 F'^{01} + \Lambda^0{}_1 \Lambda^1{}_0 F'^{10}$

$E_x = \gamma^2 E'_x + \gamma^2 (v^2/c^2) (-E'_x)$

$E_x = E'_x$

**$\mu\nu = 02$**

$F^{02} = \Lambda^0{}_\rho \Lambda^2{}_\sigma F'^{\rho\sigma} = \Lambda^0{}_0 \Lambda^2{}_2 F'^{02} + \Lambda^0{}_1 \Lambda^2{}_2 F'^{12}$

$E_y = \gamma E'_y + c\,\gamma (v/c) B'_z$

$E_y = \gamma (E'_y + v B'_z)$

**$\mu\nu = 03$**

similarly

$E_z = \gamma (E'_z - v B'_y)$

**$\mu\nu = 12$**

$F^{12} = \Lambda^1{}_\rho \Lambda^2{}_\sigma F'^{\rho\sigma}$

$= \Lambda^1{}_0 \Lambda^2{}_2 F'^{02} + \Lambda^1{}_1 \Lambda^2{}_2 F'^{12}$

$B_z = \gamma (v/c) (E_y/c) + \gamma B'_z$

$B_z = \gamma (B'_z + (v/c^2) E_y)$

**$\mu\nu = 31$**

similarly

$B_y = \gamma (B'_y - (v/c^2) E_z)$

**$\mu\nu = 23$**

$F^{23} = \Lambda^2{}_\rho \Lambda^3{}_\sigma F'^{\rho\sigma}$

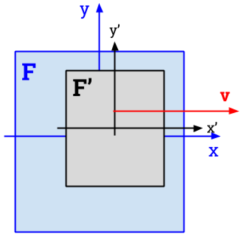
$= \Lambda^2{}_2 \Lambda^3{}_3 F'^{23}$

$B_x = B'_x$

# Tables of coordinate transformations and field transformations

Suppose reference frame ***F'*** moves with velocity **v** = v **i** with respect to frame ***F***.

Picture:




***Coordinate transformations (*** $x^{\mu} \leftarrow x'^{\mu}$ ***)***

$x^{\mu} = \Lambda^{\mu}{}_{\rho}\, x'^{\rho}$

$$\Lambda^{\mu}{}_{\rho} = \begin{matrix} \gamma & \gamma v/c & 0 & 0 \\ \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$t = \gamma\,(t' + (v/c^2)\,x')$

$x = \gamma\,(x' + v\,t')$

$y = y'$

$z = z'$

***Field transformations (*** $F^{\mu\nu} \leftarrow F'^{\mu\nu}$ ***)***

$E_x = E'_x$

$E_y = \gamma\,(E'_y + v\,B'_z)$

$E_z = \gamma\,(E'_z - v\,B'_y)$

$B_x = B'_x$

$B_y = \gamma\,(B'_y - (v/c^2)\,E_z)$

$B_z = \gamma\,(B'_z + (v/c^2)\,E_y)$

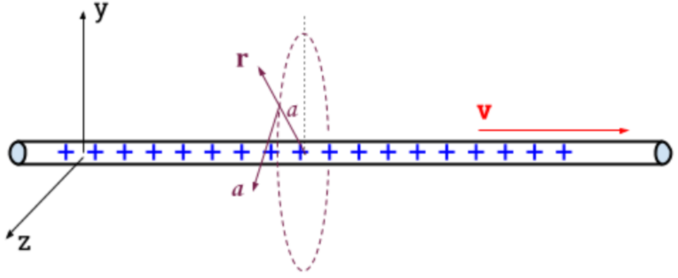
***Example.***

Consider a line of charge that is at rest in reference frame ***F'***, lying along the x' axis; charge per unit length = λ.

*Use the field transformations to calculate the electric and magnetic fields in the frame **F**.*

*NOTA BENE: the line of charge moves in the x direction relative to frame **F**.*

**Picture:**




**Solution:**

First, what are the fields **E'** and **B'**?

By Gauss's law, the electric field around a charged line points radially away from the line, with magnitude $\lambda/(2\pi\varepsilon_0 r')$ where r' is the perpendicular distance from the line.

$\mathbf{E}' = \lambda/(2\pi\varepsilon_0 r') \; \{ (y'/r')\,\mathbf{j} + (z'/r')\,\mathbf{k} \}$

On the other hand **B'** = 0 because there is no current.

The transformed fields (relative velocity = v **i**):

$E_x = E'_x = 0$

$E_y = \gamma E'_y$

$E_z = \gamma E'_z$

$\mathbf{E} = \gamma\lambda/(2\pi\varepsilon_0 r') \; \{ (y'/r')\,\mathbf{j} + (z'/r')\,\mathbf{k} \}$

$\mathbf{E} = \gamma\lambda/(2\pi\varepsilon_0 r) \; \{ (y/r)\,\mathbf{j} + (z/r)\,\mathbf{k} \}$ ;

this is the same as a line of charge with density γλ, larger than λ because of Lorentz contraction.

$B_x = B'_x = 0$

$B_y = -\gamma (v/c^2) E_z$

$B_z = \gamma (v/c^2) E_y$

$\mathbf{B} = \gamma(v/c^2)\,\lambda/(2\pi\varepsilon_0 r') \; \{ (-z'/r')\,\mathbf{j} + (y'/r')\,\mathbf{k} \}$

$\mathbf{B} = \mu_0 I/(2\pi r) \; \{ (-z/r)\,\mathbf{j} + (y/r)\,\mathbf{k} \}$ ;

this is the same as a line of current with

$I = \gamma \lambda v$, by Ampere's law.

***Next lecture --***

calculate the electric and magnetic fields of a single charge that moves with constant velocity v.