The energy-momentum flux tensor

Energy and momentum

For a ***particle***, $\xi^\mu(\tau)$,

$$p^\mu = m\, d\xi^\mu / d\tau$$
$$p^i = m\, u^i / \sqrt{1-u^2/c^2}$$
$$E = c\, p^0 = mc^2 / \sqrt{1-u^2/c^2}$$

For a ***field***, energy (and momentum) are spread throughout the field;

$$u(\mathbf{x},t) = (\varepsilon_0/2)\, E^2(\mathbf{x},t) + 1/(2\mu_0)\, B^2(\mathbf{x},t)$$
$$U = \int u\, d^3x$$

Now field energy (and momentum) needs to be expressed in terms of tensors --- to satisfy the postulate of relativity

$$T^{\mu\nu} = \frac{1}{\mu_0}\left\{ F^{\mu\rho} F^{\nu}{}_{\rho} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right\}$$

Note: $T^{\mu\nu}$ is a tensor

We'll see that $T^{\mu\nu}$ describes the energy and momentum flux.

To do: Understand the interpretation of $T^{\mu\nu}$; and express $T^{\mu\nu}$ in terms of **E** and **B.**
Recall

$$F^{\mu\nu} = \begin{array}{cc} & \nu \longrightarrow \\ \mu \downarrow & \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \end{array} \qquad g^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Symmetry properties
$g^{\mu\nu}$ is symmetric : $g^{\nu\mu} = g^{\mu\nu}$
$F^{\mu\nu}$ is antisymmetric : $F^{\nu\mu} = - F^{\mu\nu}$
$T^{\mu\nu}$ is symmetric : $T^{\nu\mu} = T^{\mu\nu}$

$$F^{\nu\rho} F^{\mu}{}_{\rho} = F^{\nu}{}_{\rho} F^{\mu\rho} = F^{\mu\rho} F^{\nu}{}_{\rho}$$

Calculate the divergence (4d) of $T^{\mu\nu}$

$$\mu_0 \frac{\partial T^{\mu\nu}}{\partial x^\nu} = \frac{\partial F^{\mu\rho}}{\partial x^\nu} F^\nu{}_\rho + F^{\mu\rho} \frac{\partial F^\nu{}_\rho}{\partial x^\nu} - \frac{1}{4} g^{\mu\nu} \left(\frac{\partial F^{\rho\sigma}}{\partial x^\nu} F_{\rho\sigma} + F^{\rho\sigma} \frac{\partial F_{\rho\sigma}}{\partial x^\nu} \right)$$

Each term is a 4-vector with index μ

- **2nd term =** $F^{\mu\rho} \left(-\mu_0 J_\rho \right)$
- **4th term = same as 3rd term**
- **'3' + '4' =** $-\frac{1}{2} g^{\mu\nu} \frac{\partial F^{\rho\sigma}}{\partial x^\nu} F_{\rho\sigma} = \frac{1}{2} F_{\alpha\beta} \frac{\partial F^{\beta\alpha}}{\partial x_\mu}$
- **1st term** = $\frac{1}{2} F_{ab}\, dF^{mb}/dx_a + \frac{1}{2} F_{ab}\, dF^{am}/dx_b$
- **'1' + '3' + '4'** =

$$\frac{1}{2} F_{\alpha\beta} \left\{ \frac{\partial F^{\beta\alpha}}{\partial x_\mu} + \frac{\partial F^{\mu\beta}}{\partial x_\alpha} + \frac{\partial F^{\alpha\mu}}{\partial x_\beta} \right\}$$

Theorem. The sum of the 3 cyclic permutations is 0 because of symmetry or because of the Maxwell's equation of the Dual Tensor.

$$\frac{\partial F^{\beta\alpha}}{\partial x_\mu} + \frac{\partial F^{\mu\beta}}{\partial x_\alpha} + \frac{\partial F^{\alpha\mu}}{\partial x_\beta}$$

$$\mu\beta\alpha \qquad \alpha\mu\beta \qquad \beta\alpha\mu$$

Proof.

Try $\mu\beta\alpha = 123$:

$$\frac{\partial B^1}{\partial x^1} + \frac{\partial B^3}{\partial x^3} + \frac{\partial B^2}{\partial x^2} = \nabla \cdot \vec{B} = 0$$

Try $\mu\beta\alpha = 0ij$:

$$-\frac{\partial F^{ij}}{\partial x^0} + \frac{\partial F^{0i}}{\partial x^j} + \frac{\partial F^{j0}}{\partial x^i}$$

$$= -\frac{1}{c} \frac{\partial B^k}{\partial t} + \frac{1}{c} \frac{\partial E^i}{\partial x^j} - \frac{1}{c} \frac{\partial E^j}{\partial x^i}$$

$$= -\frac{1}{c} \left\{ \partial \vec{B}/\partial t + \nabla \times \vec{E} \right\}^k = 0$$

Try $\mu\beta\alpha = 00i$:

$$\frac{\partial F^{0i}}{\partial x_0} + \frac{\partial F^{00}}{\partial x_i} + \frac{\partial F^{i0}}{\partial x_0} = 0.$$

Result: $\mu_0\, \partial T^{\mu\nu} / dx^\nu = -\mu_0 F^{\mu\rho} J_\rho$

$$\partial T^{\mu\nu}/dx^\nu = -F^{\mu\rho}J_\rho$$

The interpretation

This is the continuity equation for energy and momentum conservation.

Proof. First, the case $\mu = 0$ ...

$$\frac{\partial T^{0\nu}}{\partial x^\nu} = -F^{00}J_0 - F^{0i}J_i$$

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^i} = -\frac{E^i}{c}J^i$$

$$\frac{\partial u}{\partial t} + \nabla\cdot\vec{S} = -\vec{E}\cdot\vec{J}$$

$$\left(T^{00} = u \quad \text{and} \quad cT^{0i} = S^i\right)$$

NB: **E.J** = the work done by **E** on the charged particles, per unit time: per unit volume:

$dW = \mathbf{F} \,.\, d\mathbf{x} = (\rho\, dV)\, \mathbf{E} \,.\, \mathbf{v}\, dt = \mathbf{E} \,.\, \mathbf{J}\, dt\, dV$

Thus we must have

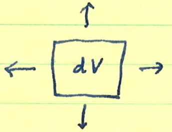
$T^{00} = u$ (*energy density*)

$T^{0i} = S^i/c$ (*energy flux*)

then

$\partial u/\partial t + \nabla\cdot\mathbf{S} + \mathbf{E}\cdot\mathbf{J} = 0$

which expresses the conservation of energy.



$$\frac{\delta U}{\delta t} + \frac{\delta W}{\delta t} = -\vec{S}\cdot\delta\vec{A}$$

Now write u and **S** in terms of **E** and **B**

$$u = T^{00} = \frac{1}{\mu_0}\left\{\frac{E^i}{c}\frac{E^i}{c} + \frac{1}{4}\left(-2E^2/c^2 + 2B^2\right)\right\}$$

$$u = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$$

$$S^i = \frac{c}{\mu_0}F^{0\rho}F^{i}{}_{\rho} = \frac{c}{\mu_0}\frac{E^j}{c}\epsilon_{ijk}B^k$$

$$\vec{S} = \vec{E}\times\vec{B}/\mu_0$$

$$\partial T^{\mu\nu} / dx^{\nu} = - F^{\mu\rho} J_{\rho}$$

Density and flux of momentum

We have , for $\mu = i$ (= 1, 2, 3)

$\partial T^{i\nu} / dx^{\nu} = - F^{i\rho} J_{\rho}$

LHS = $\partial T^{i0} / dx^0 + \partial T^{ij} / dx^j$

$= (1/c^2)\, \partial S^i / dt + (\boldsymbol{\nabla} \cdot \mathbf{T})^i$

RHS = $- F^{i0} J_0 - F^{ij} J_j$

$= - (-E^i / c)\, (-c\, \rho) - \varepsilon_{ijk} J^j B^k$

$= - \{ \rho\, E + \mathbf{J} \times \mathbf{B} \}^i$

Thus

$$\partial \boldsymbol{\Pi}_{\text{fields}} / \partial t + \{ \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \} = - \boldsymbol{\nabla} \cdot \mathbf{T}$$

where $\boldsymbol{\Pi}_{\text{fields}} = \mathbf{S}/c^2$

This equation is the continuity equation for momentum conservation.

(i) $\mathbf{S} = \mathbf{E} \times \mathbf{B} / \mu_0$ = energy flux

$\mathbf{S}/c^2$ = momentum density = $\boldsymbol{\pi}_{\text{fields}}$

(Check units!)

(ii) $\boldsymbol{\nabla} \cdot \mathbf{T}$ = divergence of the momentum flux

$T^{ij} =$

(iii) The force per unit volume on particles = $\{ \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \} = \partial \boldsymbol{\Pi}_{\text{particles}} / \partial t$

(momentum density of the particles).

E. M. fields carry momentum, and the total momentum (fields + particles) is conserved at every point in space.

$$\partial (\boldsymbol{\Pi}_{\text{fields}} + \boldsymbol{\Pi}_{\text{particles}}) / \partial t = - \boldsymbol{\nabla} \cdot \mathbf{T}$$