## Electromagnetic Waves in a Conductor

e.g., metals, plasma

Recall electrostatics: the electric field in an ideal conductor is 0. But now we deal with time-dependent fields.

We'll consider a simple model:

$$\mathbf{J}(\mathbf{x},t) = \sigma\, \mathbf{E}(\mathbf{x},t) \qquad \text{(Ohm's law)}$$

i.e., an ideal ohmic conductor

***Realistic*** conductors:

- $\sigma$ depends on $\omega$
- at high frequencies, electron dynamics becomes important
- matter is atomic, not a continuum

But this simple model is a good starting point.

## Macroscopic Equations

$$\nabla \cdot \mathbf{E} = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\,\partial\mathbf{B}/\partial t \quad \text{and} \quad \nabla \times \mathbf{B} = \underbrace{\mu\sigma\,\mathbf{E}}_{\text{conduction current}} + \underbrace{\mu\varepsilon\,\partial\mathbf{E}/\partial t}_{\text{displacement current}}$$

We seek a wave-like solution with angular frequency $\omega$,

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0\, e^{i(kx - \omega t)}$$

i.e., an ideal plane wave propagating in the x direction.

The <u>Real Part</u> is implied; but we'll use complex functions to simplify the calculation, and take the real part at the end of the calculation!

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0\, e^{i(kx - \omega t)}$$

$\nabla \cdot \mathbf{E} = 0$ implies $\mathbf{e}_x \cdot \mathbf{E}_0 = 0$ ;

i.e., transverse oscillations of **E**.

$\nabla \times \mathbf{E} = -\ \partial \mathbf{B}/\partial t$ implies

$$ik\, \mathbf{e}_x \times \mathbf{E}_0\, e^{i(kx - \omega t)} = -\ \partial \mathbf{B}/\partial t$$

Thus $\mathbf{B}(\mathbf{x},t) = (k/\omega)\, \mathbf{e}_x \times \mathbf{E}_0\, e^{i(kx - \omega t)}$

I.e., $\mathbf{E}_0$, $\mathbf{B}_0$, **and k** form an orthogonal triad.

$\nabla \cdot \mathbf{B} = 0$ implies $\mathbf{e}_x \cdot \mathbf{B}_0 = 0$ ;

i.e., transverse oscillations of **B**.

(already true)

$\nabla \times \mathbf{B} = \mu\sigma\, \mathbf{E} + \mu\varepsilon\, \partial \mathbf{E}/\partial t$ implies

$$(k/\omega)\, (ik\, \mathbf{e}_x) \times (\mathbf{e}_x \times \mathbf{E}_0)\, e^{i(kx - \omega t)}$$

$$= -\,(ik^2/\omega)\, \mathbf{E}_0\, e^{i(kx - \omega t)}$$

$$= \{\ \mu\sigma + \mu\varepsilon\,(-i\omega)\}\, \mathbf{E}_0\, e^{i(kx - \omega t)}$$

Thus ... $k^2 = \mu\varepsilon\, \omega^2 + i\, \mu\sigma\omega$

(called: the *dispersion relation* )

*So,* ***k is complex.***

*Eventually we'll take the real part* ***of the fields****,*

*but not yet.*

$k^2 = \mu\varepsilon\,\omega^2 + i\,\mu\sigma\omega$

Write $k = \kappa_1 + i\,\kappa_2$ ($\kappa_1$ and $\kappa_2$ are real)

Then $\kappa_1^2 - \kappa_2^2 = \mu\varepsilon\,\omega^2$ and $2\,\kappa_1\,\kappa_2 = \mu\sigma\omega$

Solution is

$$\kappa_{1,2} = \omega\sqrt{\mu\varepsilon/2}\,\left[\sqrt{1 + \sigma^2/(\varepsilon\omega)^2} \pm 1\right]^{1/2}$$

Checks

- In a dielectric ( $\sigma = 0$ )

  $\{\kappa_1 ; \kappa_2\} = \{\omega\sqrt{\mu\varepsilon} = \omega/v \;;\; 0\}$ *ok*

- $\kappa_1^2 - \kappa_2^2 = \omega^2\,(\mu\varepsilon/2)\,.\,2 = \mu\varepsilon\,\omega^2$ *ok*

- $2\,\kappa_1\,\kappa_2 = 2\;\omega^2\,(\mu\varepsilon/2)\,.\,\sigma/(\varepsilon\omega) = \mu\sigma\omega$ *ok*

**Attenuation of the wave** : $k = \kappa_1 + i\,\kappa_2$

$$\mathbf{E}(\mathbf{x},t) = \mathrm{Re}\;\mathbf{E}_0\,e^{\,i\,(kx - \omega t)}$$

$$= \mathbf{E}_0\,e^{-\kappa 2\,x}\;\cos(\kappa_1 x - \omega t)$$

$$\mathbf{B}(\mathbf{x},t) = \mathrm{Re}\;\{\;(\mathbf{k}/\omega)\;\mathbf{e}_x \times \mathbf{E}_0\,e^{\,i\,(kx - \omega t)}\;\}$$

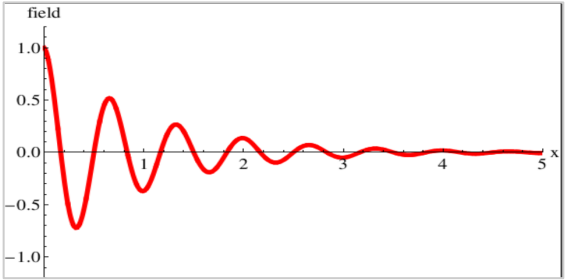
$$= \mathbf{e}_x \times \mathbf{E}_0/\omega\;\;e^{-\kappa 2\,x}$$

$$\{\kappa_1\cos(\kappa_1 x - \omega t) - \kappa_2\sin(\kappa_1 x - \omega t)\}$$

$$= \mathbf{e}_x \times \mathbf{E}_0\;\;(|\mathbf{k}|/\omega)\;\;e^{-\kappa 2\,x}$$

$$\cos(\kappa_1 x - \omega t + \varphi)$$

Attenuation factor : $e^{-\kappa 2\,x}$




Phase shift of **B** : $\tan\varphi = \kappa_2/\kappa_1$

**The case of "good conductors" ;**
**i.e., when σ is "large"**

$$\kappa_{1,2} = \omega \sqrt{\mu\varepsilon/2}\,[\sqrt{1+\sigma^2/(\varepsilon\,\omega)^2} \pm 1]^{1/2}$$

The σ dependence only occurs in the form σ/(ε ω).

- Therefore, "large σ" means σ >> ε ω.
- In this case, $\kappa_1 = \kappa_2$ = SQRT [ μσω /2]

- Wavelength = $2\pi / \kappa_1$ =
  (2π v/ω) SQRT[ 2 ε ω/σ ]
  wavelength without conductivity
  x a small factor

- Attenuation length
  $\delta = 1/\kappa_2$ = SQRT[ 2/ (μ σ ω) ]
  "skin depth"

★ δ is small for metals;
★ e.g., δ ~ 100 atomic layers would be typical for I.R. or visible light.
★ METALS ARE OPAQUE.

- <u>Exercise</u>

  Show :

  the phase shift ≈ 45 degrees;

  and v $B_0$ / $E_0$ (which is equal to 1 without conductivity) is >> 1 for good conductors.

next time:
Reflection of light from a conducting surface.
Why are metals shiny?