Dispersion of electromagnetic waves in simple dielectrics

"Dispersion" means that optical properties depend on frequency.
(Colors are dispersed in a rainbow.)

To understand why, we need to consider atomic dynamics; i.e., electron dynamics.

The best theory is quantum mechanics, but we'll consider a simpler classical model due to Lorentz.

A classical model for frequency dependence

The electrons experience a harmonic force,

$$\mathbf{F} = -e\,\mathbf{E}(\mathbf{x},t) \propto \exp\{-i\,\omega t\}$$

this **x** is the electron position

So, the equation of motion for an electron is

$$m\,x'' = -k\,x - \gamma\,x' - e\,E_0 \exp\{-i\,\omega t\}$$

restoring force; the electron is bound in an atom

resistive force; damping, or energy transfer

the electric force exerted by the e. m. wave; with frequency ω

You will remember this example of dynamics from PHY 321: the damped harmonic oscillator.

The damped driven oscillator

$$m\, x'' = -k\, x - \gamma\, x' - e\, E_0 \exp\{-i\,\omega t\}$$

The STEADY STATE SOLUTION is

$$x(t) = x_0 \exp\{-i\,\omega t\}\,;$$

i.e., the electron oscillates at the driving frequency ω. The only question is, what is the amplitude of oscillation?

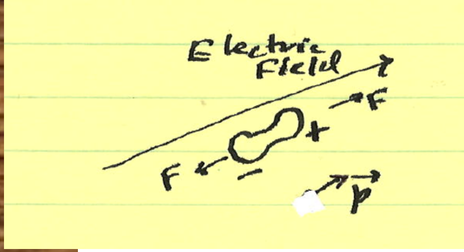
$$(k - m\,\omega^2 - i\,\omega\,\gamma)\,x_0 = -\,e\,E_0$$

$$x_0 = \frac{-\,e\,E_0}{k - m\,\omega^2 - i\,\omega\,\gamma}$$

This should give a reasonable picture of the frequency dependence for optical frequencies or below. However, at high frequencies (photons of UV light or X-rays) a quantum theory would be necessary for more accuracy.

Dispersion in a dielectric

i.e., an insulator;
all the charge is *bound* charge.
Picture



Dipole moment

$$\mathbf{p} = -\,e\,\mathbf{x} = \alpha\,\mathbf{E}$$

α = the atomic polarizability

$$\alpha = \frac{e^2}{k - m\,\omega^2 - i\,\omega\,\gamma}$$

But α is complex!
What does that imply?

Remember: We must take the ***Real Part*** of complex ***E*** and complex ***x*** for the physical quantities.

The Clausius - Mossotti formula

From Chapter 6, Eq. 6-39 relates the atomic parameter (α) to the macroscopic parameter (ε)

$$\frac{\varepsilon}{\varepsilon_0} = \frac{3\varepsilon_0 + 2\,\alpha\,v}{3\varepsilon_0 - \alpha\,v}$$

where v = atomic density (# atoms /m^3)

If $\alpha v << \varepsilon_0$ then we may approximate

$$\varepsilon / \varepsilon_0 \approx 1 + \alpha v / \varepsilon_0 \;;$$

$$= 1 + (ve^2 / \varepsilon_0) / (k - m\omega^2 - i\,\omega\gamma)$$

== complex

== frequency dependent

== resonant

<u>Propagation of an electromagnetic wave</u>

$\mathbf{E}(\mathbf{x},t) = E_0\, \mathbf{e_y} \exp\{ i\,(Kx - \omega t)\}$

ideal transverse plane wave

Apply Maxwell's equations $\Rightarrow$

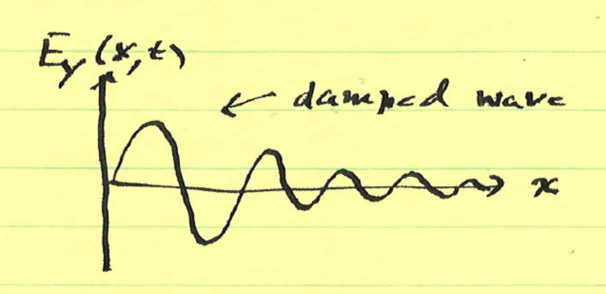
$K = \omega / v_{phase} = \omega\sqrt{\varepsilon\mu_0}$, as usual.

But now ε is complex and frequency dependent. So, we must write

$K = \kappa_1 + i\,\kappa_2$

$\Rightarrow \quad \mathbf{E}(x,t) = E_0\, \mathbf{e_y} \exp\{-\kappa_2 t\} \exp\{ i\,(\kappa_1 x - \omega t)\}$

Picture



ABSORPTION

The energy absorbtion length $\delta = \frac{1}{2\kappa_2}$

DISPERSION

The index of refraction, dependent on ω,

$$n = \frac{c}{v_{phase}} = \frac{c\kappa_1}{\omega} = \frac{c\kappa_1(\omega)}{\omega}$$

RESONANCE

We have $\kappa_1 + i\kappa_2 = \omega\sqrt{\epsilon\mu_0}$

$\epsilon/\epsilon_0 = 1 + \nu\alpha/\epsilon_0$ assuming $\nu\alpha \ll \epsilon_0$

$\alpha = \dfrac{e^2}{m(\omega_0^2 - \omega^2) - i\omega\gamma}$ w/ $\omega_0 = \sqrt{K/m}$ the resonant frequency

A little bit of complex analysis —

$$\frac{c\kappa_1}{\omega} = 1 + \frac{\nu}{2\epsilon_0} \operatorname{Re}\alpha = 1 + \frac{\nu}{2\epsilon_0}\frac{e^2}{m}\frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + (\omega\gamma/m)^2}$$

$$\frac{c\kappa_2}{\omega} = \frac{\nu}{2\epsilon_0} \operatorname{Im}\alpha = \frac{\nu}{2\epsilon_0}\frac{e^2}{m}\frac{\omega\gamma/m}{(\omega_0^2-\omega^2)^2 + (\omega\gamma/m)^2}$$

Results

Index of refraction

$$n = \frac{c k_1}{\omega} = 1 + \frac{\nu e^2}{2 \epsilon_0 m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\omega \gamma / m)^2}$$

Inverse attenuation length

$$\delta^{-1} \approx 2 k_2 = \frac{\nu e^2}{\epsilon_0 m^2 c} \frac{\gamma \omega^2}{(\omega_0^2 - \omega^2)^2 + (\omega \gamma / m)^2}$$

Do you see the <u>*resonance?*</u>

ω_0 = SQRT (K/m) = the resonant frequency

What is the resonant width?

___ more next time