PROPAGATION OF ELECTROMAGNETIC WAVES IN A DILUTE PLASMA

Plasma -- an ionized gas.
Electric current is carried by electrons and ionized atoms. Because the electrons are much less massive than the ions, the current is dominated by the electron motion.
($a_e = F/m_e >> F/m_{ion} = a_{ion}$)

Use the classical electron model for the current.

The conductivity is complex!
What does that imply about the propagation of electromagnetic waves? (We previously studied the effect of conductivity with real **σ**.)

The free electrons in a plasma obey the equation of motion　　/1/

$$m \frac{d^2\vec{x}}{dt^2} = -\gamma \frac{d\vec{x}}{dt} - e\vec{E}_0 e^{-i\omega t}$$

damping mechanism　　electric force

The steady-state solution is

$$\vec{x}(t) = \frac{e}{m\omega^2 + i\omega\gamma} \vec{E}_0 e^{-i\omega t}$$

the electron position oscillates at the driving frequency.

and the velocity is

$$\vec{v}(t) = \frac{d\vec{x}}{dt} = \frac{-i\omega e}{m\omega^2 + i\omega\gamma} \vec{E}(t)$$

Current density

Let ν_e = electron density (# electrons/m^3)

Then $\vec{J} = -e\vec{v}\,\nu_e$　　units: $C \frac{m}{s} m^{-3} = A/m^2$ ✓

So $\vec{J} = \sigma\vec{E}$　(like Ohm's law)

where $\sigma(\omega) = \dfrac{i\omega e^2 \nu_e}{m\omega^2 + i\omega\gamma}$

Case of a Dilute Plasma

It's not simply that $\mathbf{n_e}$ is small, because $\boldsymbol{\sigma}$ is proportional to $\mathbf{n_e}$. What we mean is that *collisions are rare*. Then $\boldsymbol{\gamma}$ is small, i.e., $\mathbf{i\omega\gamma} \ll \mathbf{m\omega^2}$.

Energy sloshes back and forth between the fields and the current, but there is no energy loss.

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So, we'll take $\sigma(\omega) = \dfrac{i e^2 \nu_e}{m\omega}$

purely imaginary σ

$\boxed{\text{There is no energy loss.}}$

Recall $\dfrac{dP}{dV} = \vec{J}\cdot\vec{E}$ = power transfer from the electromagnetic field to the electrons.

$\vec{E} = \mathrm{Re}\, \vec{E}_0\, e^{i(kx-\omega t)} = \vec{E}_0 \cos(kx-\omega t)$

$\vec{J} = \mathrm{Re}\, \sigma\vec{E} = -\dfrac{e^2 \nu_e}{m\omega} \vec{E}_0 \sin(kx-\omega t)$

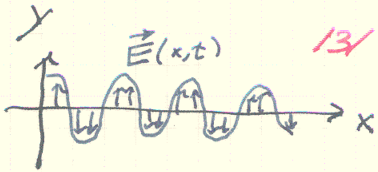
The electric field and current oscillates 90 degrees out of phase;

$$\vec{E}\cdot\vec{J} = -\frac{e^2\nu_e}{m\omega} E_0^2 \underbrace{\cos(kx-\omega t)\sin(kx-\omega t)}_{\text{average value is } 0.}$$

The Dispersion Relation

i.e., the relation between k and ω …

The Dispersion Relation



$$\vec{E} = E_0\,\hat{j}\,e^{i(kx-\omega t)}$$

$$\vec{B} = \frac{kE_0}{\omega}\,\hat{k}\,e^{i(kx-\omega t)}$$

$$\underbrace{\nabla\times\vec{B}}_{\textstyle\hookrightarrow\ \mu_0\sigma\vec{E} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}} = \frac{kE_0}{\omega}\,ik\;\hat{i}\times\hat{k}\;e^{i(kx-\omega t)} = -\frac{ik^2E_0}{\omega}\,\hat{j}\,e^{i(kx-\omega t)}$$

$$\mu_0\sigma\vec{E} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = (\mu_0\sigma - \mu_0\epsilon_0\,i\omega)\,E_0\,\hat{j}\,e^{i(kx-\omega t)}$$

$$-\frac{ik^2}{\omega} = -\frac{i\omega}{c^2} + \mu_0\frac{ie^2\nu_e}{m\omega}$$

$$\boxed{k^2 = \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2} \quad \text{where} \quad \omega_p^2 = \frac{e^2\nu_e}{m\epsilon_0}}$$

Plasma frequency $\omega_p = \sqrt{\dfrac{e^2\nu_e}{m\epsilon_0}}$; Dispersion relation $k^2 = \dfrac{\omega^2-\omega_p^2}{c^2}$.

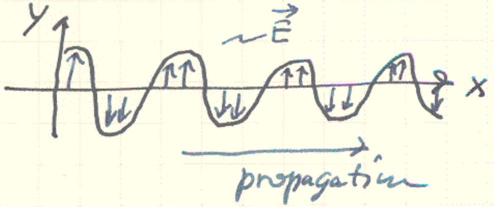
Propagation of Electromagnetic Waves

First, high frequencies …

High frequencies $\vec{E} = E_0 \hat{j} e^{i(kx-\omega t)}$ (real part) /3/

If $\omega > \omega_p$ then k is real and the wave propagates with constant amplitude.

$$\vec{E} = E_0 \hat{j} \cos(kx-\omega t)$$



Dispersion The phase velocity is

$$v_{phase} = \frac{\omega}{k} = \frac{\omega}{c^{-1}\sqrt{\omega^2-\omega_p^2}}$$ depends on frequency.

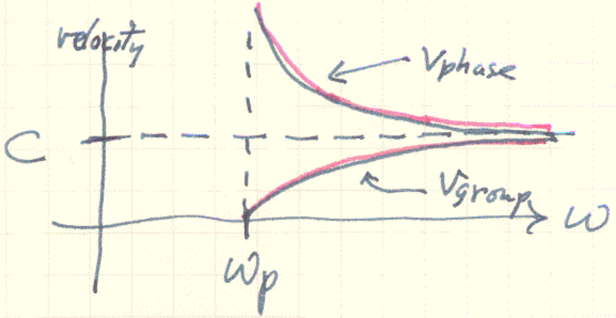
Propagation of finite electromagnetic waves, i.e., waves with finite longitudinal extent (as opposed to ideal plane waves). *(pulses or wave packets)*

$$v_{phase} = \frac{\omega}{k} = \frac{\omega c}{\sqrt{\omega^2 - \omega_p^2}} > c \qquad \boxed{k^2 = \frac{\omega^2 - \omega_p^2}{c^2}} \quad /4/$$

$$v_{group} = \frac{d\omega}{dk} = \frac{\sqrt{\omega^2 - \omega_p^2}}{\omega}\, c < c \quad \leftarrow \text{ i.e. } c^2 / v_{phase}$$

$$dk^2 = 2k\,dk = \frac{d\omega^2}{c^2} = \frac{2\omega\, d\omega}{c^2} \Rightarrow \frac{d\omega}{dk} = \frac{c^2 k}{\omega}$$

$$v_{phase}\, v_{group} = c^2$$



Propagation of Electromagnetic Waves

Second, low frequencies …

<u>Low frequencies</u>

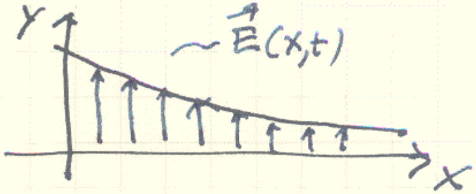
$$K = \frac{\sqrt{\omega_p^2-\omega^2}}{c} \qquad /4/$$

If $\omega<\omega_p$ the k is purely imaginary;

$k = iK$ where K is real $\Longrightarrow$ attenuation, no propagation.

$$\vec{E} = E_0\,\hat{j}\,e^{i(iKx-\omega t)} = E_0\,\hat{j}\,e^{-Kx}\,e^{-i\omega t}$$

real part $= E_0\,\hat{j}\,e^{-Kx}\cos\omega t$



In <u>electrostatics</u> an electric field does not penetrate into a conductor (e.g., a plasma); the electrons move to <u>screen</u> the electric field.

Exercise. For low frequencies ($\boldsymbol{\omega} < \boldsymbol{\omega}_p$) calculate the <u>attenuation length</u> .

The Ionosphere

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The ionosphere $\quad \omega_p = \sqrt{\dfrac{e^2 \nu_e}{m \epsilon_0}}$

$\nu_e = 10^{11}$ electrons/m^3 $\quad \leftarrow$ but variable!

$\omega_p = 2 \times 10^7 \; s^{-1} = 20$ MHz

Attenuation length $\quad \dfrac{c}{\omega_p} = \dfrac{3 \times 10^8 \; m/s}{2 \times 10^7 \; s^{-1}} = 15 \; m$

Consider a radio wave moving toward the ionosphere.
If $\omega < \omega_p$ the wave cannot propagate,
so it reflects from the ionosphere

AM radio $\quad 0.55$ MHz $< f < 1.60$ MHz

3.3 MHz $< \omega < 9.6$ MHz

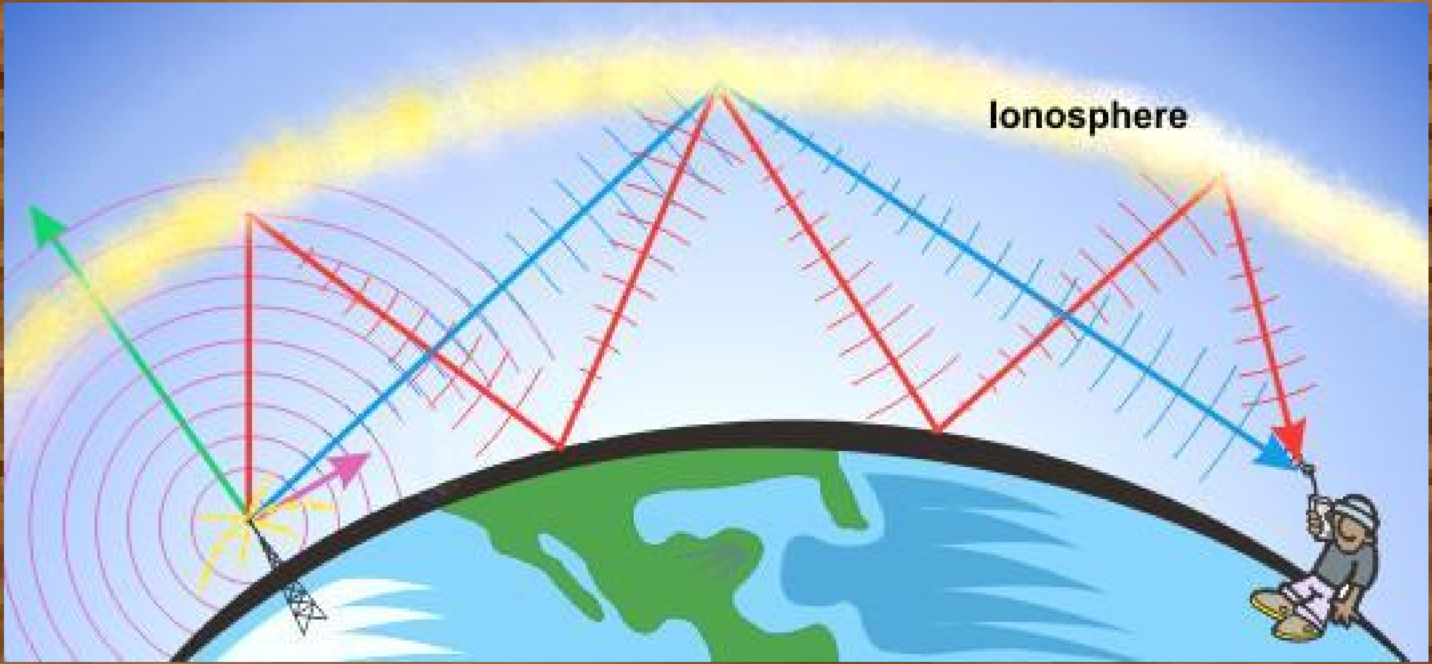
Thus, $\omega < \omega_p$; implies reflection from the ionosphere

//Interesting History — Oliver Heaviside and Guglielmo Marconi//

FM radio $\quad \omega \sim 100$ MHz

Thus, $\omega > \omega_p$; no reflection

These radio waves just pass through the ionosphere.



Exercise –
What is the attenuation length in the plasma?