Radiation of Electromagnetic Waves

We need...

Field Equations - to relate charge, current and fields

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon 0 \; ; \qquad \nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\,\partial\mathbf{B}/\partial t \; ; \quad \nabla\times\mathbf{B} = \mu_0\,\mathbf{J} + \mu_0\,\varepsilon_0\,\partial\mathbf{E}/\partial t$$

Potential functions - to simplify the solution of the equations

$$\mathbf{B} = \nabla\times\mathbf{A}\,; \qquad \mathbf{E} = -\,\partial\mathbf{A}/\partial t - \nabla V$$

Lorentz gauge condition

$$\nabla\cdot\mathbf{A} = -\,1/c^2\,\partial V/\partial t \qquad \text{(gauge fixing)}$$

Then $\mathbf{A}(\mathbf{x},t)$ and $V(\mathbf{x},t)$ obey the inhomogeneous wave equation,

$$-\,\nabla^2 V + (1/c^2)\,\partial^2 V/\partial t^2 = \rho/\varepsilon_0$$

$$-\,\nabla^2\mathbf{A} + (1/c^2)\,\partial^2\mathbf{A}/\partial t^2 = \mu_0\,\mathbf{J}$$

The d'Alembertian

$$\square \equiv 1/c^2 \;\; \partial^2/\partial t^2 - \nabla^2 \,,$$

a differential operator ;

$$\square\, V = \rho/\varepsilon_0 \;\; ; \qquad \square\,\mathbf{A} = \mu_0\,\mathbf{J} \;\; .$$

We can solve for V and **A** (in terms of ρ and **J**) if we know the Green's function of □; i.e., $G(\mathbf{x},t;\,\mathbf{x}',t')$.

Green's functions

Suppose we have a linear differential operator $\mathcal{D}$ and an inhomogeneous equation

$$\mathcal{D} f = \sigma \qquad (1)$$

f(x) : the function we want to determine;

σ(x) : a known function; the source of f;

x : the coordinates, which may have multiple components.

The Green's function G(x;x') is defined by

$$\mathcal{D} G = \delta^n(x-x')\,; \qquad (2)$$

i.e., G(x;x') is the function for a point source at x'.

Then $$f(x) = \int G(x;x')\, \sigma(x')\, d^nx'. \qquad (3)$$

The problem is solved; i.e., it is reduced to integration.

Proof#1. Equation (3) is just the superposition principle.

Proof#2.

$$\mathcal{D} f(x) = \int \mathcal{D} G(x;x')\ \sigma(x')\ d^nx'$$

$$= \int \delta^n(x-x')\ \sigma(x')\ d^nx'$$

$$= \sigma(x) \qquad \text{QED}$$

Example 1 : Electrostatics

$$-\nabla^2 V = \rho/\varepsilon_0$$

$$-\nabla^2 \left(1/\ 4\pi|\mathbf{x} - \mathbf{x}'| \right) = \delta^3(\mathbf{x}-\mathbf{x}')$$

$$V(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')\, d^3x'}{4\pi\varepsilon_0|\mathbf{x} - \mathbf{x}'|} = \int \frac{dQ}{4\pi\varepsilon_0\, r}$$

"reduced to an integral"

The Green's function of $-\nabla^2$ is $1/4\pi|\vec{x}-\vec{x}'|$.

Example 2 : harmonic time dependence

<u>Harmonic time dependence</u>

Suppose $\rho(\vec{x},t) = \tilde{\rho}(\vec{x})\, e^{-i\omega t}$

Then $V(\vec{x},t) = \tilde{V}(\vec{x})\, e^{-i\omega t}$

and $-\nabla^2 \tilde{V} - \frac{\omega^2}{c^2}\tilde{V} = \tilde{\rho}/\epsilon_0$

<u>Theorem</u> The Green's function of $-\nabla^2 - k^2$

is $\dfrac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} \equiv \tilde{G}(\vec{x},\vec{x}')$

<u>Proof</u> We need to prove : $-\nabla^2\tilde{G} - k^2\tilde{G} = \delta^3(\vec{x}-\vec{x}')$

W.L.O.G., let $\vec{x}' = 0$.

- $\nabla^2 \dfrac{e^{ikr}}{r} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial}{\partial r}\right)\dfrac{e^{ikr}}{r}$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{ikr\, e^{ikr} - e^{ikr}\right\} = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{(ikr-1)e^{ikr}\right\}$$

$$= \left\{\frac{ik}{r^2} + \frac{ik}{r^2}(ikr-1)\right\}e^{ikr} = -k^2\frac{e^{ikr}}{r}$$

Thus $(\nabla^2 + k^2)\dfrac{e^{ikr}}{r} = 0$ (for $r \neq 0$)

- Now consider $\int_V (\nabla^2 + k^2)\dfrac{e^{ikr}}{r}\, dV$ (V: a sphere volume, centered at the origin $(r=0)$)

Gauss's theorem

$$= \oint_S \hat{r}\cdot\nabla\left(\frac{e^{ikr}}{r}\right) dA + k^2\int \frac{e^{ikr}}{r}\, dV$$

$\int_0^R \frac{1}{r}\, 4\pi r^2 dr = 4\pi\frac{R^2}{2}$

$$\xrightarrow[R\to 0]{} \frac{-1}{R^2}\, 4\pi R^2 + k^2\, 2\pi R^2$$

$$\xrightarrow[R\to 0]{} -4\pi$$

$$\therefore (\nabla^2 + k^2)\frac{e^{ikr}}{r} = -4\pi\,\delta^3(\vec{x})$$

QED

Result

If $\rho(\mathbf{x},t) = \tilde{\rho}(\mathbf{x})\, e^{-i\omega t}$

then $V(\mathbf{x},t) = \tilde{V}(\mathbf{x})\, e^{-i\omega t}$ where

$$\tilde{V}(\mathbf{x}) = \int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi\varepsilon_0 |\mathbf{x}-\mathbf{x}'|}\, \tilde{\rho}(\mathbf{x}')\, d^3x'$$

Note: $k = \omega / c$

General time dependence

Arbitrary $\rho(\mathbf{x},t)$ …

We can write $\rho(\mathbf{x},t)$ as a Fourier integral; then use Eq. (4) for each frequency; and then apply the superposition principle to the set of frequencies.

$$\rho(\vec{x},t) = \int_{-\infty}^{\infty} \tilde{\rho}(\vec{x},\omega)\, e^{-i\omega t}\, d\omega$$

$$V(\vec{x},t) = \int_{-\infty}^{\infty} \tilde{V}(\vec{x},\omega)\, e^{-i\omega t}\, d\omega$$

$$= \frac{1}{\epsilon_0}\int_{-\infty}^{\infty} \int \frac{e^{i\omega|\vec{x}-\vec{x}'|/c}}{4\pi|\vec{x}-\vec{x}'|}\, \tilde{\rho}(\vec{x},\omega)\, d^3x'\, e^{-i\omega t}\, d\omega$$

$$= \frac{1}{\epsilon_0}\int \frac{\rho(\vec{x}, t-|\vec{x}-\vec{x}'|/c)}{4\pi|\vec{x}-\vec{x}'|}\, d^3x'$$

The retarded time $t' = t - |\mathbf{x}-\mathbf{x}'|/c$

$$\int_{-\infty}^{\infty} \tilde{\rho}(\mathbf{x}',\omega)\, \underbrace{e^{i\omega|\mathbf{x}-\mathbf{x}'|/c}\, e^{-i\omega t}}_{e^{-i\omega(t-|\mathbf{x}-\mathbf{x}'|/c)}}\, d\omega = \rho(\mathbf{x}',t')$$

The retarded potentials

$$V(\vec{x},t) = \frac{1}{\epsilon_0}\int \frac{\rho(\vec{x}', t - |\vec{x}-\vec{x}'|/c)}{4\pi\,|\vec{x}-\vec{x}'|}\, d^3x'$$

$$\vec{A}(\vec{x},t) = \mu_0 \int \frac{\vec{J}(\vec{x}', t - |\vec{x}-\vec{x}'|/c)}{4\pi\,|\vec{x}-\vec{x}'|}\, d^3x'$$

Note that V(**x**,t) and **A(x**,t) depend on ρ(**x’,**t’) and **J(x’,**t’),
where t’ is the *earlier* time
t’ = t - | **x-x’** | /c.
The time delay | x - x’ |/c is equal to the time that it takes for light to travel from **x’** to **x**.
In other words, if the source suddenly changes at x’, the potential at x will not change until the light-travel-time later.

The Green’s function of □

The Green's function of $\Box$

$$\Box G = \delta^3(\vec{x}-\vec{x}')\,\delta(t-t')\ ;$$

i.e., G is a time-dependent Green's function

$G = G(\vec{x},t\ ;\ \vec{x}',t')$.

Theorem $\quad G = \dfrac{\delta(\tau - r/c)}{4\pi r}$ where $r = |\vec{x}-\vec{x}'|$, $\tau = t - t'$

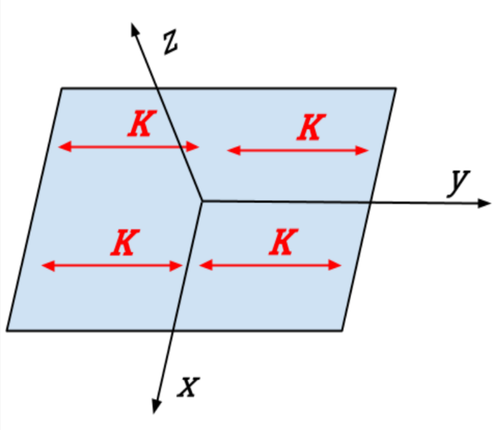
Proof Because then

$$V(\vec{x},t) = \int G(\vec{x}t\ ;\ \vec{x}'t')\,\frac{\rho(\vec{x}',t')}{\epsilon_0}\, d^3x'\,dt'$$

$$= \frac{1}{\epsilon_0}\int \frac{\rho(\vec{x}', t - |\vec{x}-\vec{x}'|/c)}{4\pi\,|\vec{x}-\vec{x}'|}\, d^3x'$$

SAME AS ABOVE!

Example - generating a plane wave



The horizontal plane (z = 0) carries an electric current with surface current density

$$\mathbf{K}(x,y,t) = K_0\,\mathbf{j}\,e^{-i\,\omega t}$$

(A) Determine the vector potential $\mathbf{A} = A_y(z,t)\,\mathbf{j}$.

(B) Determine the magnetic field $\mathbf{B}(z,t)$.

{Nota Bene: By translation invariance, **A** and **B** do not depend on x or y.}

$$\vec{A}(\vec{x},t) = \mu_0 \int \frac{\vec{J}(\vec{x}',t')\,d^3x'}{4\pi\,|\vec{x}-\vec{x}'|}$$

$t' = t - |\vec{x}-\vec{x}'|/c$

or, rather, $$\mu_0 \int \frac{\vec{K}(\vec{x}',t')\,d^2x'}{4\pi\sqrt{(x-x')^2+(y-y')^2+z^2}}$$

set $x=0$ and $y=0$ because the answer does not depend on x or y

$$= \mu_0 \frac{K_0\,\hat{j}}{4\pi}\int_0^\infty e^{-i\omega t}\,e^{i\omega\sqrt{r'^2+z^2}/c}\,\frac{2\pi r'\,dr'}{\sqrt{r'^2+z^2}}$$

$$= \frac{\mu_0 K_0}{2}\,\hat{j}\,e^{-i\omega t}\,\frac{c}{i\omega}\,e^{i\omega\sqrt{r'^2+z^2}/c}\Big]_{r'=0}^{r'=\infty}$$

$$= \frac{\mu_0 K_0 c}{\omega}\,\hat{j}\,e^{-i\omega t}\left\{ i e^{i\omega z/c} + \text{an ignorable "constant" term} \right\}$$

don't need to worry about this; artifact of infinite planar source

$$\vec{A}(\vec{x},t) = \frac{\mu_0 K_0}{k}\,\hat{j}\,i\,e^{i(kz-\omega t)} \quad \text{where } k = \frac{\omega}{c}$$

↳ a plane wave propagating in z direction.

Exercise: Calculate $\vec{E}$, $\vec{B}$ and $\vec{S}$.