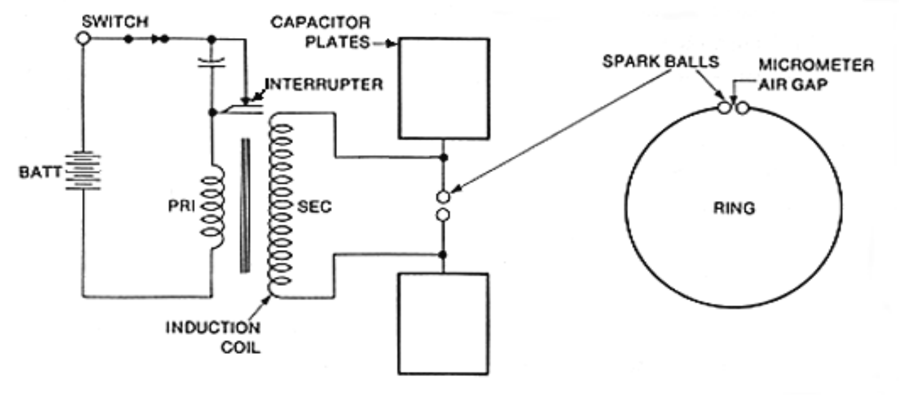
Radiation of Electromagnetic Waves

Heinrich Hertz - 1888



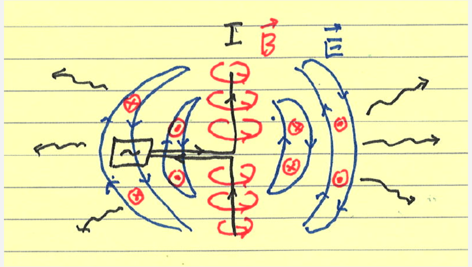
The English mathematical physicist, Sir Oliver Heaviside, said in 1891, "Three years ago, electromagnetic waves were nowhere. Shortly afterward, they were everywhere."

Electric Dipole Radiation

Given sources ($\rho(x,t)$ and/or $\mathbf{J}(\mathbf{x},t)$) what is the radiation?

To make plane waves, we would need infinite planar sources. That may be an interesting academic exercise, but not realistic. More important -- **finite sources make spherical waves.**

Picture



Using the Lorentz gauge,

$$\mathbf{A}(\mathbf{x},t) = \mu_0 \int \frac{\mathbf{J}(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)\, d^3x'}{4\pi|\mathbf{x} - \mathbf{x}'|}$$

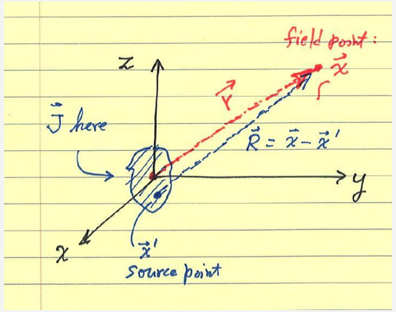
{ We might also have $\rho(x,t)$ but not necessarily. In any case $\nabla\cdot\mathbf{J} = -\partial\rho/\partial t$}

The radiation fields

Calculate the asymptotic fields, i.e., propagating away from the sources.

We are interested in the fields as $r \rightarrow \infty$; i.e., fields far from the sources. There, we have only fields in empty space, so these asymptotic fields must approach wave solutions of the Maxwell equations in empty space.

Picture



$$\vec{R} = |\vec{x} - \vec{x}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta} \quad \text{(exact)}$$

$$R \approx r \quad \text{lowest approximation}$$

$$R \approx r - r'\cos\theta + O\left(\frac{r'^2}{r}\right) \quad \text{improved approx.}$$

The exact equation depends on $R = |\,\mathbf{x} - \mathbf{x}'\,|$.

But the lowest order approximation is good enough to determine the "radiation fields" ; so we have $\mathbf{A}(\mathbf{x},t) \sim \mu_0/(4\pi\, r) \int \mathbf{J}(\mathbf{x}', t - r/c)\, d^3x'$

Theorem.

Let $\mathbf{p}(t')$ be the *electric dipole moment* of the source at the time t'. Then

$$\vec{A}(\vec{x},t) \sim \frac{\mu_0}{4\pi r} \int \vec{J}(\vec{x}';t')\, d^3x' = \frac{\mu_0}{4\pi r} \frac{d\vec{p}}{dt'}$$

Proof.

{Temporarily drop the prime (') on all the coordinates. Put it back at the end.}

Note that

$$\int \nabla\cdot (x_i \mathbf{J})\, d^3x = 0\ ,$$

by Gauss's theorem because the function $x_i \mathbf{J}(\mathbf{x},t)$ is 0 at infinity. So

$$0 = \int \{\, J_i + x_i\, \nabla\cdot\mathbf{J}\,)\, d^3x$$

$$0 = \int (J_i\, d^3x \; - \; x_i\, \partial\rho/\partial t\,)\, d^3x$$

The definition of the dipole moment of a charge distribution is $\mathbf{p} = \int \mathbf{x}\, \rho\, d^3x$. Therefore

$$\int J_i\, d^3x = dp_i/dt\ . \qquad \text{Q.E.D.}$$

$$\mathbf{A}(\mathbf{x},t) \sim \frac{\mu_0}{4\pi r} \frac{d\mathbf{p}}{dt'}$$

evaluated at t' = t - r/c ;
R = | x - x' | ;   r = | x |

The asymptotic $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$

$$\vec{B} = \nabla \times \vec{A} \sim \frac{\mu_0}{4\pi}\left\{ \frac{-\hat{r}}{r^2} \times \frac{d\vec{p}}{dt'} + \frac{1}{r}\left(\frac{-\hat{r}}{c}\right) \times \frac{d^2\vec{p}}{dt'^2} \right\}$$

$$\nabla r = \hat{r} \; ; \quad \nabla t' = -\hat{r}/c$$

This term is negligible : $O(1/r^2)$
Neglect $O(1/r^2)$ in $\vec{E}$ and $\vec{B}$ as $r \to \infty$.

$$\vec{B} \sim \frac{-\mu_0}{4\pi r c} \hat{r} \times \frac{d^2\vec{p}}{dt'^2} \equiv \vec{B}_{rad}.$$

This is the magnetic part of the radiation field.
Note $\vec{B}_{rad} \propto 1/r$.

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V$$ We would need V

Simpler : use the field equation

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$ where $\vec{J} = 0$

$$\therefore \frac{\partial \vec{E}}{\partial t} = c^2 \nabla \times \vec{B} \sim \frac{-\mu_0 c}{4\pi} \nabla \times \left(\frac{\hat{r}}{r} \times \vec{p}''\right)$$

$$\vec{E} \approx \frac{-\mu_0 c}{4\pi} \nabla \times \left(\frac{\hat{r}}{r} \times \vec{p}'\right)$$

∇ on $\hat{r}/r$ is $O(1/r^2)$; neglect as $r \to \infty$

∇ on $\vec{p}'$ is $\hat{r}\frac{\partial}{\partial r} = -\frac{\hat{r}}{c}\frac{\partial}{\partial t'}$

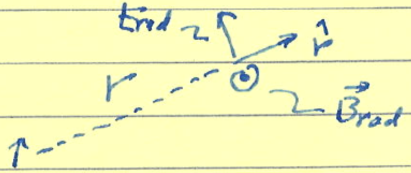
$(t' = t - r/c)$

$$\vec{E} \approx -\frac{\mu_0 c}{4\pi}\left(\frac{-\hat{r}}{c}\right) \times \left(\frac{\hat{r}}{r} \times \vec{p}''\right)$$

<u>Result</u> $$\vec{E} \sim \frac{\mu_0}{4\pi r} \hat{r} \times (\hat{r} \times \vec{p}'') \equiv \vec{E}_{rad}$$

Note that $\vec{E}_{rad} = -c\,\hat{r}\times\vec{B}_{rad}$;

so $\hat{r}$, $\vec{E}_{rad}$ and $\vec{B}_{rad}$ form an orthogonal triad;



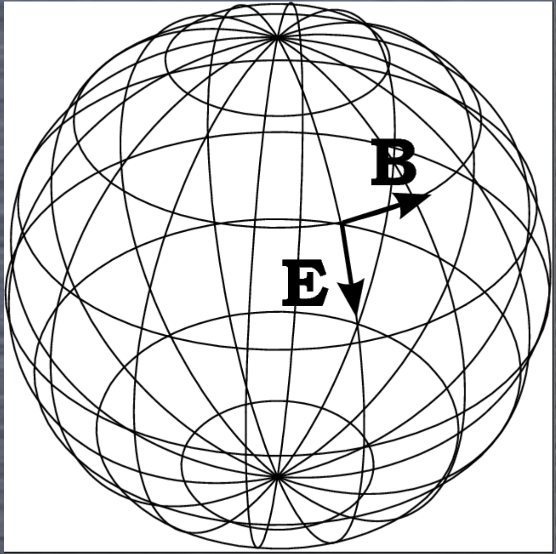
and $\frac{B_{rad}}{E_{rad}} = \frac{1}{c}$;

like a plane wave.

$$\vec{S}_{rad} = \frac{1}{\mu_0}\vec{E}_{rad}\times\vec{B}_{rad} = \frac{\mu_0\hat{r}}{(4\pi r)^2 c}\left[|\vec{p}\,''|^2 - (p_r'')^2\right]$$

Power radiated $P = \int \vec{S}_{rad}\cdot\hat{r}\,r^2\,d\Omega$ (at time t')

$$P = \frac{\mu_0}{4\pi c}\frac{1}{4\pi}\int\left[|\vec{p}\,''|^2 - (p_r'')^2\right]d\Omega$$



15.2.1 The Hertzian Dipole

$$\vec{p}(t) = \hat{e}_z \, p_0 \cos \omega t$$

15.2.2 Atomic Transitions

15.2.3 Magnetic Dipole Radiation

15.2.4 Complete Fields of a Hertzian Dipole

15.3 The Half Wave Linear Antenna

15.4 Radiation from a Point Charge