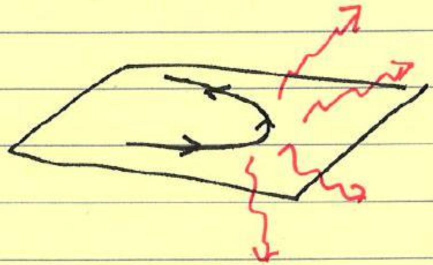
Radiation from a point charge

If a charged particle accelerates then it radiates electromagnetic waves.



To calculate : the radiation power

Source functions

$$\rho(\vec{x},t) = q\, \delta^3(\vec{x} - \vec{x}_q(t))$$

$$\vec{J}(\vec{x},t) = q\, \vec{v}(t)\, \delta^3(\vec{x} - \vec{x}_q(t))$$

$$\vec{v} = d\vec{x}_q/dt$$

The continuity equation is obeyed:

$$\frac{\partial \rho}{\partial t} = q\left[\nabla \delta^3(\vec{x}-\vec{x}_q)\right] \cdot \left(-\frac{d\vec{x}_q}{dt}\right)$$

$$= -\nabla \cdot \left(q\, \vec{v}\, \delta^3(\vec{x}-\vec{x}_q)\right) = -\nabla \cdot \vec{J} \quad \checkmark$$

Potentials

$$\vec{A}(\vec{x},t) = \mu_0 \int \vec{J}(\vec{x}', t_r') \frac{d^3x'}{4\pi|\vec{x}-\vec{x}'|}$$

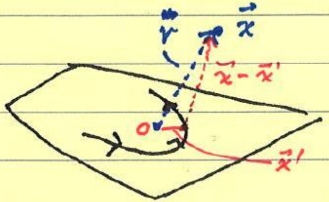
$t - |\vec{x}-\vec{x}'|/c$

Asymptotically,

$$\vec{A}(\vec{x},t) \sim \frac{\mu_0}{4\pi r}\int \vec{J}(\vec{x}', t')\, d^3x'$$

$t' \sim t - r/c$

$r = |\vec{x}-\vec{x}'|$



$$\vec{J}(\vec{x}', t') = q\,\vec{v}(t')\,\delta^3(\vec{x}' - \vec{x}_q(t'))$$

A subtle point: if $|\vec{v}| \lll c$ then we can approximate $\int \vec{J}(\vec{x}, t')\, d^3x' \approx q\,\vec{v}(t_r)$

where $t_r = t - |\vec{x} - \vec{x}_q(t_r)|/c$

$= t - R/c$

Result For nonrelativistic motion of q, ★

$$\vec{A}(\vec{x},t) \sim \frac{\mu_0}{4\pi R}\, q\,\vec{v}(t_r)$$

$\vec{a} = d\vec{v}/dt_r$ acceleration

$$\vec{B}(\vec{x},t) \sim -\frac{\mu_0 q}{4\pi c}\,\frac{\hat{R}\times\vec{a}}{R} \equiv \vec{B}_{rad}.$$

$$\vec{E}(\vec{x},t) \sim c\,\vec{B}_{rad}\times\hat{R} \equiv \vec{E}_{rad}.$$

where $\vec{R} = \vec{x} - \vec{x}_q(t_r)$ and $t_r = t - R/c$.

★ See pages 594–596 for exact result; i.e., valid even if $|\vec{v}|$ is not $\ll c$.

Energy Flux and Larmor's formula

$$\vec{S}_{rad.} = \frac{1}{\mu_0} \vec{E}_{rad} \times \vec{B}_{rad}$$

$$= \frac{1}{\mu_0}\left(\frac{\mu_0 q}{4\pi}\right)^2 \frac{1}{cR^2}\left[(\hat{R}\times\vec{a})\times\hat{R}\right]\times(\hat{R}\times\vec{a})$$

$$= \frac{q^2}{16\pi^2\epsilon_0 c^3}\frac{1}{R^2}\underbrace{(\hat{R}\times\vec{a})^2}\,\hat{R}$$

← direction is $\hat{R}$, directed away from the pos. of q at t_r

$$= \underbrace{\epsilon_{ijk}\hat{R}_j a_k \epsilon_{ilm}}\hat{R}_l a_m$$

$$= \left[\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right]\hat{R}_j \cdot a_k \hat{R}_l a_m$$

$$= a^2 - (\hat{R}\cdot\vec{a})^2$$

$$= a^2(1-\cos^2\theta)$$

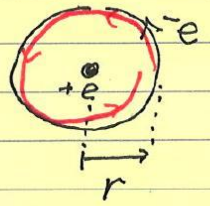
Power radiated at time $t_r = t - R/c$

$$P = \int_{\substack{\text{sphere}\\ \text{of radius } R}} \vec{S}_{rad}\cdot\hat{R}\,R^2\,d\Omega$$

$$= \frac{q^2 a^2}{16\pi^2\epsilon_0 c^3}\underbrace{\int(1-\cos^2\theta)\sin\theta\,d\theta}_{\int_{-1}^{1}(1-u^2)\,du \;=\; 2-2/3 \;=\; 4/3}\underbrace{d\phi}_{2\pi}$$

$$P = \frac{1}{4\pi\epsilon_0}\frac{2q^2a^2}{3c^3}$$ (Larmor, 1897)

Instability of the classical atom



H atom

Classically, the electron would lose energy by radiation, and spiral into the nucleus.

$\Rightarrow$ lifetime of the atom $\sim 10^{-11}$ s. ($r \sim 10^{-10}$ m)

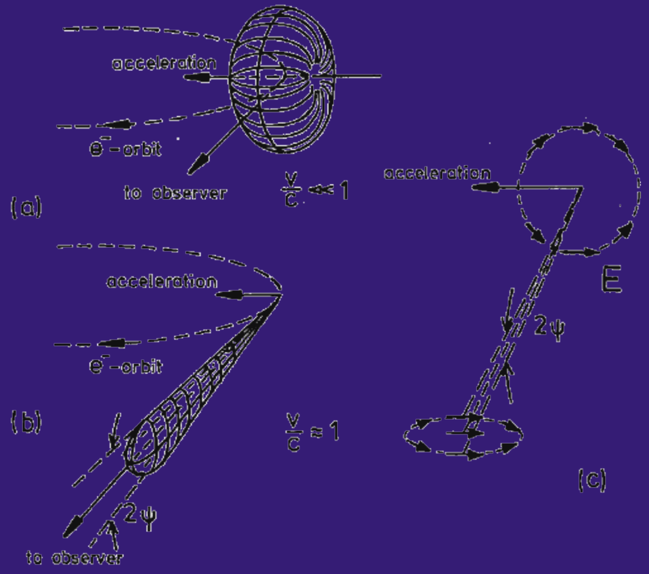
Estimates: $\quad \dfrac{mv^2}{r} = \dfrac{e^2}{4\pi\epsilon_0 r^2}$

$$a = \frac{v^2}{r} = \frac{e^2}{4\pi\epsilon_0 m r^2} \sim 2.5 \times 10^{22} \text{ m/s}^2$$

$$P = \frac{e^2}{4\pi\epsilon_0}\,\frac{2a^2}{3c^3} \sim 3.6 \times 10^{-9} \text{ W} \sim 2.3 \times 10^{10} \text{ eV/s}$$

$$\text{lifetime} \sim \frac{\text{energy}}{\text{power}} \sim 4 \times 10^{-10} \text{ sec.}$$ (energy $\sim$ 10 eV)

More "accurately": See pages 586–588.



Another example: Synchrotron Radiation
by electrons moving on helical orbits in a strong magnetic field.